Loan-to-Value Ratio, Chonsei-to-Price Ratio, and the Optimal Housing Decision

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Abstract

The unique housing rental system in the Republic of Korea, known as *chonsei*, involves both housing rental transactions and financial transaction of large-scale deposits exchanged between individuals. Moreover, this chonsei system is where the Korean government takes particular interest in housing stability for low-income individuals, formulating and implementing various support policies. In this study, we present, for the first time, a rigorous continuous-time optimal portfolio selection model that includes both chonsei housing selection and home purchase decisions for a household. Subsequently, we derive closed-form solutions and conduct comparative statics on various factors influencing household decisions. In particular, we analyze the impact of variables determined in both the housing and financial markets, as well as those determined by government policies, while distinguishing between them.

Keywords: chonsei; loan-to-value ratio; chonsei-to-price ratio; mortgage loan; guarantee for chonsei loan; optimal stopping time; martingale method

1. Introduction

Most governments around the world make significant efforts to ensure housing stability for their citizens. This is achieved by providing affordable housing options for low-income individuals and families, either through subsidized rental housing or homeownership assistance programs such as mortgages. The Republic of Korea (hereafter, Korea) has a unique housing rental system called chonsei, which is not found in other countries. The Korean government is actively working to promote this system.

In a chonsei contract, the tenant pays a large lump sum deposit, typically ranging from 50% to 70% of the property's value, to the landlord at the beginning of the lease. This deposit is then returned to the tenant in full at the end of the lease term. The ratio of the deposit to the property value is known as the *chonsei-to-price* ratio. The chonsei system is often referred to as a *bridge to homeownership* because it falls somewhere between monthly rent and homeownership, allowing tenants to gradually save up enough money to purchase a home of their own. Unlike monthly rent, tenants do not pay any regular fees to the landlord for using the property. Instead, the monthly rent is effectively

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deducted from the interest earned on the key money that the landlord holds during the lease term. This results in a significant difference between chonsei and monthly rent. The following Figure 1 illustrates the chonsei system (Ahn and Ryu, 2024):

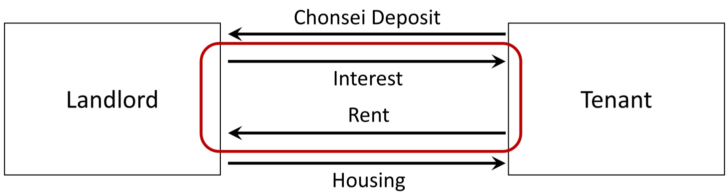


Figure 1: The Structure of Chonsei Contract

The chonsei system is generally preferred by tenants over monthly rent due to its long-term lease and lower interest on the deposit compared to the monthly rent. In Korea, monthly rent contracts typically have a term of one year, while chonsei contracts typically have a term of two years. Additionally, recent legislation allows tenants to extend their contracts for an additional two years under the same conditions if they wish. The interest on the deposit is generally lower than the monthly rent. This means that tenants can save money over the long term by choosing chonsei instead of monthly rent.

Recognizing the importance of chonsei in stabilizing the housing market, the Korean government has implemented various policies to promote the system, including the guarantee for chonsei loans The guarantee for chonsei loans is designed to assist tenants who have difficulty securing the large upfront payment of deposit required for chonsei. Through this program, public institutions provide guarantees that allow tenants to obtain chonsei loans from commercial banks. Even tenants with limited credit can obtain chonsei loans through the guarantee. The effective interest rates on chonsei loans through the guarantee, that is, the sum of the guarantee fee and the interest on the loan, are significantly lower not only than those of personal loans based on the tenant's creditworthiness, but also than the total monthly rent payments over the same period. The guarantee for chonsei loans plays a crucial role in promoting housing stability for low-income individuals and families in Korea. It provides an essential financial support system that enables tenants to access affordable housing options and achieve long-term housing stability.

In this study, we aim to analyze the impact of the government's guarantee for chonsei loans on the tenant's housing service choices between renting, that is, chonsei and purchasing a home using a microeconomic portfolio selection model. Specifically, we seek to examine how the availability of low-interest loans relative to the tenant's creditworthiness, facilitated by the guarantee, affects the tenant's decision between leasing and buying a home. Furthermore, we will compare and analyze the effects of the loan-to-value (hereafter, LTV) ratio, applicable when directly purchasing a home, and the chonsei-to-price ratio. When opting for a chonsei, the tenant is granted a loan limit equivalent to the chonsei-to-price ratio, while choosing to purchase results in a loan limit based on the LTV ratio. Therefore, comparing the influence of these two variables, which play a significant role in determining the actual loan limit, on the tenant's decision is meaningful. Additionally, we will compare the interest rates of chonsei loans and mortgage loans, examining the landlord's housing selection from a cost perspective.

The detailed illustration of the model of this study is as follows: This study deals with the tenant's housing, consumption, and investment selection model on the continuous-time approach. The tenant chooses between living in a house through chonsei or purchasing one, while simultaneously selecting the size of the house of residence. However, the purpose of home purchase is restricted to primary residence rather than housing investment, assuming that relatively low-income tenants primarily aim for stable residence rather than housing investment. To simplify the model, during the chonsei period, the tenant selects a housing size from relatively small and large houses, while when purchasing one, the tenant has complete freedom to choose the size. It's worth mentioning that although it's a continuous-time model, the housing service selection is effectively modeled on a discrete-time basis, while stock and bond investments, on continuous-time basis. In reality, housing selection (excluding housing investment) occurs very rarely due to transaction costs and low liquidity, making it impractical to model it in continuous-time as stock and bond investments.

The subsequent sections of this study are outlined below. The contribution of relevant literature review and our study to the existing literature is provided in Section 2. In Section 3, we introduce an optimal consumption and investment problem involving housing purchase and chonsei-switching decisions within a continuous time framework. This is done under a theoretical framework that encompasses both the housing and financial markets. In Section 4, we provide a comprehensive derivation process for the martingale approach and variational inequality, leading to the closed-form solutions for optimal consumption, investment, housing state, purchase timing, and purchased property size. In Section 5, we concentrate on the impacts of changes in the parameter values associated with deposit loans and mortgage loan services on optimal strategies. Through comparative static analysis, we numerically elucidate the fluctuations in the optimal consumption, investment, chonsei-switching wealth boundary, purchasing wealth boundary, and size of the purchased house. The conclusion is presented in Section 6, while detailed proofs are provided in the Appendix.

2. Related Work

Existing literatures predominantly focus on how a choice for a fixed house, whether rented or owned, impacts non-durable consumption and stock investments. However, studies addressing housing mobility and its effects on consumption (excluding housing consumption) and investment decisions are notably scarce. Similarly, few have comprehensively analyzed the optimization of demand for housing consumption and housing investment choice. The demand for the housing consumption may stem from either rented or owned properties. Although Cocco (2005) and Yao and Zhang (2005) have indicated that housing investments could diminish participation in the stock market and Piazzesi et al. (2007) developed an asset pricing model that includes housing services, highlighting a predictive relationship between the housing market and stock returns, these studies did not examine the optimal timing for purchasing house. Sinai and Souleles (2005) investigated the role of homeownership in mitigating the volatility of rental prices. Such theoretical models and empirical evidence from Sinai and Souleles (2005) showed how owning property could alleviate the uncertainties tied to renting. Building on the empirical findings of Sinai and Souleles (2005) and the theoretical frameworks of Li and Ahn (2022) and Li et al. (2024a), our study introduces a consumption-housing-investment model that aligns with the trend of wealth accumulation for individuals in the realistic economy and examines the implications of housing mobility. We aim to investigate how variations in rental behaviors and the optimal homebuying choices influence individual consumption patterns and stock investment decisions. This model seeks to bring to a focus on understanding the dynamic interplay between changing housing status and financial portfolio adjustments over time.

Contrary to the typical rental model, we specifically take into account the deposit-only model, only available to the real estate market of South Korea. Kim (2013) provided an empirical analysis indicating that the chonsei system emerged as a mortgage product in South Korea's financially repressed environment, with the aim of renting out properties. The research in Ronald and Jin (2015) delved into the shifts occurring within Korea's chonsei sector, taking into consideration alterations in economic landscapes and household compositions. By the exploration of default risk, Park and Pyun (2020) examined the factors influencing the chonsei, combined deposit-rent, and purely monthly rent leases in South Korea's rental market. In the study conducted by Ahn and Ryu (2024), an analysis of a continuous-time model incorporating conversions between chonsei and monthly rent was undertaken, focusing on the lessors' viewpoint.

Moreover, our study contributes to the literature on mortgage loans. Chambers et al. (2009) investigated the impact of loan configuration on both borrower's mortgage contract choices and the overall economy, introducing a quantitative equilibrium framework for mortgage selection. Analyzing the effects of adjustable-rate mortgage adjustments on consumer actions, Di Maggio et al. (2017) uncovered that notable decreases in mortgage payments resulted in considerable upticks in vehicle acquisitions, especially among borrowers with lower incomes, underscoring the importance of household financial conditions and the adaptability of mortgage contracts in the transmission of monetary policy. Liu (2023) presented a novel methodology by incorporating household transitions internally and then explored the effects of fixed-rate mortgage loans on rental and owner-occupied housing markets throughout various life phases. The results indicated that the duration of mortgages affects homeownership rates, whereas the mortgage interest rate and initial payment ratio have minimal impact. Li et al. (2024a) introduced an optimal transition framework for individuals shifting from renting to homeownership, incorporating the decision of mortgage selection. In Table 1, we conduct a comparative analysis of the primary related literature.

3. The economy

3.1. Housing Market

In this study, we investigate the consumption and stock investment behaviors across the life cycle of individuals, transitioning from renting to buying, rooted in their housing service demands. We specifically examine the Korean-

Table 1: An analysis of prominent literature on the topics of housing service, housing investment, mortgage loan, and closed-form solution.

	Individual		Optimal purchasing	Mortgage loan (LTV ratio)	Closed-form solution
	Renter	Owner			
Yao and Zhang (2005)	✓	✓	×	✓	×
Hu (2005)	✓	✓	×	✓	×
Li and Ahn (2022)	✓	✓	✓	×	✓
Ahn and Ryu (2024)	×	✓	×	×	✓
This paper	✓	✓	✓	✓	✓

specific rental model for chonsei system. Preceding homebuying, individuals are presumed to possess the liberty to change residences. This allows that they can opt to transition from one chonsei house to the other of varying size at any point, with this chonsei-switching behavior being reversible. We represent the values of two chonsei houses using h_1 and h_2, satisfying $h_1 < h_2$, and $\xi \cdot h_i$ denotes the deposit for chonsei, where ξ represents the chonsei-to-price ratio and $i \in \{1, 2\}$. Given the high deposit requirement for chonsei lease, individuals can access the deposit borrowing services from the bank and repay the interest on the deposit loan at the interest rate δ.

As wealth accumulates, individuals have the opportunity to purchase a house through a mortgage, deciding both the timing and the size of the purchase. This decision to buy a home is irreversible, implying that once an individual makes a purchase, they are assumed to remain in that home indefinitely. We denote the purchase time as τ, h_3 represents the total value of the purchased house, L is the loan-to-value (LTV) ratio, and ϵ denotes the mortgage rate. The convertible chonsei and purchasing behaviors reflect the mobility of housing. Table 2 details the housing types and associated costs before and after the purchase.

Table 2: The notation of house type, house value, and cost.

House type	Chonsei		Purchasing
	H_1	H_2	H_3
House value	h_1	h_2	h_3
Cost	$h_1\xi\delta$	$h_2\xi\delta$	$h_3L\epsilon$

3.2. *Financial Market*

The economy involves trading two primary assets, disregarding transaction costs. The first asset, denoted as S_t^0, represents a risk-free asset like a money market account, which accrues at a risk-free interest rate $r > 0$. The second asset, described as S_t, represents a risky one such as a stock, adhering to a geometric Brownian motion process characterized by a return rate $\mu > 0$ and volatility $\sigma > 0$. In an economic setting where $\epsilon > r$, the findings from Li et al. (2024a) indicate negligible differences in the consumption and investment behaviors of individuals compared to the scenario where $\epsilon < r$. Therefore, our focus is directed towards instances where $\epsilon < r$. The dynamic processes of these two assets can be delineated as follows:

$$
\begin{aligned}
\frac{\mathrm{d}S_t^0}{S_t^0} &= r\mathrm{d}t, \\
\frac{\mathrm{d}S_t}{S_t} &= \mu\mathrm{d}t + \sigma\mathrm{d}B_t,
\end{aligned}
$$

where B_t represents a standard Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Consequently, an individual earning a fixed labor income has the following wealth level

$$
dX_t = \left[rX_t + (\mu - r)\pi_t - c_t + I - \xi\delta\left(h_1 \cdot \mathbf{1}_{\{\Xi_t = H_1\}} + h_2 \cdot \mathbf{1}_{\{\Xi_t = H_2\}}\right)_{\{0\leqslant t<\tau\}} - h_3 L\epsilon \cdot \mathbf{1}_{\{t\geq\tau\}}\right] dt + \sigma\pi_t dB_t, \qquad X_0 = x, \tag{3.1}
$$

with the natural constraint

$$
X_t > -\frac{I - h_1\xi\delta}{r}\mathbf{1}_{\{0\leqslant t<\tau\}} - \frac{I - h_3 L\epsilon}{r}\mathbf{1}_{\{\tau\leqslant t\}}.
$$

During the rental period, an individual's wealth level increases at the rate of $(rX - c + I - h_1\xi\delta \cdot \mathbf{1}_{\{\Xi=H_1\}} - h_2\xi\delta \cdot \mathbf{1}_{\{\Xi=H_2\}})$, while after buying a house, the wealth level grows at the rate of $(rX - c + I - h_3 L\epsilon)$. Here, the $\mathcal{F}_t$-measurable processes of consumption c_t and investment amount π_t satisfy the conditions for $\int_0^\infty c_t \mathrm{d}t < \infty$ and $\int_0^\infty \pi_t^2 \mathrm{d}t < \infty$.

3.3. Maximization Problem

Individuals can derive utility from both housing consumption and investment, as well as from the consumption of non-durable goods. We utilize the Cobb-Douglas utility function to describe these preferences, i.e.,

$$
u_i(c_t, h_i) \equiv \frac{1}{\alpha}\frac{\left(c_t^\alpha h_i^{1-\alpha}\right)^{1-\gamma^*}}{1-\gamma^*}, \quad \text{with } \gamma^* \in (0,1)\cup(1,\infty),\ \alpha \in (0,1),\ \text{and } i \in \{1,2\}, \tag{3.2}
$$

where γ^* denotes the coefficient of relative risk aversion and the elasticity of preference on non-durable consumption is represented by α. The utility function after the timing of purchase is expressed as $u_3(c_t, h_3) \equiv \frac{1}{\alpha}\frac{\left(\kappa c_t^\alpha h_3^{1-\alpha}\right)^{1-\gamma^*}}{1-\gamma^*}$. Here, κ is the motive for homebuying. For simplicity in calculations, we consider $\gamma \equiv 1 - \alpha(1-\gamma^*)$. Consequently, the utility function in (3.2) can be reformulated as

$$
\begin{cases}
u_i(c_t, h_i) = h_i^{\gamma-\gamma^*}\frac{c_t^{1-\gamma}}{1-\gamma}, & \text{for } 0 \leqslant t < \tau, \\
u_3(c_t, h_3) = \kappa^{\frac{1-\gamma}{\alpha}} h_3^{\gamma-\gamma^*}\frac{c_t^{1-\gamma}}{1-\gamma}, & \text{for } t \geq \tau.
\end{cases}
$$

We employ the job-switching framework proposed by Li et al. (2024b) to illustrate the chonsei-switching settings. The configuration for the optimal purchasing model remains in line with that introduced in Li and Ahn (2022) and Li et al. (2024a). Then, the optimal consumption-housing-portfolio problem, incorporating controls for consumption, investment, purchasing timing, and the selected property, can be stated in the following problem.

Problem 1 (Primal problem)**.** *The individual aims to maximize the expected utility derived from consumption, housing, and investment as follows:*

$$V(x) \quad = \quad \sup_{(c,\pi,\Xi,h_3,\tau)\in\mathcal{A}(x)} \tilde{V}(x; c, \pi, \Xi, h_3, \tau), \tag{3.3}$$

subject to the dynamic budget constraint (3.1)*, where*

$$\begin{aligned}\tilde{V}(x; c, \pi, \Xi, h_3, \tau) \quad &\equiv \quad \mathbb{E}\Bigg[\int_0^\infty e^{-\beta t}\Bigg\{\Bigg(\frac{1}{\alpha}\frac{\left(c_t^\alpha h_2^{1-\alpha}\right)^{1-\gamma^*}}{1-\gamma^*}\mathbf{1}_{\{\Xi_t=H_2\}} + \frac{1}{\alpha}\frac{\left(c_t^\alpha h_1^{1-\alpha}\right)^{1-\gamma^*}}{1-\gamma^*}\mathbf{1}_{\{\Xi_t=H_1\}}\Bigg)\mathbf{1}_{\{0\leqslant t<\tau\}} + \frac{1}{\alpha}\frac{\left(\kappa c_t^\alpha h_3^{1-\alpha}\right)^{1-\gamma^*}}{1-\gamma^*}\mathbf{1}_{\{t\geqslant\tau\}}\Bigg\}\mathrm{d}t\Bigg] \\ &= \quad \mathbb{E}\Bigg[\int_0^\tau e^{-\beta t}\left(h_2^{\gamma-\gamma^*}\frac{c_t^{1-\gamma}}{1-\gamma}\mathbf{1}_{\{\Xi_t=H_2\}} + h_1^{\gamma-\gamma^*}\frac{c_t^{1-\gamma}}{1-\gamma}\mathbf{1}_{\{\Xi_t=H_1\}}\right)\mathrm{d}t + e^{-\beta\tau}\hat{V}(x_\tau)\Bigg],\end{aligned}$$

and the value function after homebuying is defined as

$$\hat{V}(x) \equiv \sup_{(c,\pi)}\mathbb{E}\left[\int_0^\infty e^{-\beta t}\kappa^{\frac{1-\gamma}{\alpha}}h_3^{\gamma-\gamma^*}\frac{c_t^{1-\gamma}}{1-\gamma}\mathrm{d}t\right] = \frac{\kappa^{\frac{1-\gamma}{\alpha}}h_3^{\gamma-\gamma^*}}{(1-\gamma)K^\gamma}\left(x + \frac{I - h_3 L\epsilon}{r}\right)^{1-\gamma}.$$

Here, $\beta > 0$ is denoted as the subjective discount rate, and $K \equiv r + \frac{\beta-r}{\gamma} + \frac{\gamma-1}{2\gamma^2}\theta^2 > 0$ is the Merton constant.

It is worth noting that during the rental phase, individuals derive only housing consumption services from the property, as they possess only the right to use the chonsei house without ownership. Despite the risk of an increase in the chonsei deposit due to rising property prices, individuals cannot benefit from the chonsei house. Conversely, after purchasing a house, individuals can both enjoy the housing consumption services and reap housing investment benefits from the property.

4. Optimal Strategies

4.1. Martingale Approach

We apply the martingale method introduced in Karatzas et al. (1991), Li and Ahn (2022), Ahn and Ryu (2024), Li et al. (2024a) and Li et al. (2024b). Subsequently, Problem 1 mentioned above is reformulated into its corresponding dual problem. This transformation significantly contributes to providing a concise closed-form solution for both the value function and the optimal strategies.

Let's define the discounted process and the exponential martingale process as follows:

$$\zeta_t \equiv \exp\{-rt\}, \quad \mathcal{Z}_t \equiv \exp\left\{-\frac{1}{2}\theta^2 t - \theta B_t\right\}.$$

$\theta = \frac{\mu-r}{\sigma} > 0$ is the stock market price of risk. $\mathbb{Q}(A) \equiv \mathbb{E}[\mathcal{Z}_T\mathbf{1}_A]$ with $A \in \mathcal{F}_T$ is the risk-neutral measure, generated by $\mathcal{Z}_t$. After clarifying $\tilde{B}_t \equiv B_t + \theta t$, Girsanov's theorem allows us to rewrite the corresponding wealth process under the equivalent martingale measure $\mathbb{Q}$ as follows:

$$dX_t = \left[rX_t - c_t + I - \xi\delta\left(h_1\cdot\mathbf{1}_{\{\Xi_t=H_1\}} + h_2\cdot\mathbf{1}_{\{\Xi_t=H_2\}}\right)_{\{0\leqslant t<\tau\}} - h_3 L\epsilon\cdot\mathbf{1}_{\{t\geq\tau\}}\right]dt + \sigma\pi_t\mathrm{d}\tilde{B}_t.$$

Hence, by looking into the proof provided in Li et al. (2024b), we can employ a similar approach to identify the subsequent static budget constraint

$$\mathbb{E}\left[\int_0^\tau \mathcal{H}_t\left(c_t - I + \xi\delta\left(h_1 \cdot \mathbf{1}_{\{\Xi_t = H_1\}} + h_2 \cdot \mathbf{1}_{\{\Xi_t = H_2\}}\right)\right)\mathrm{d}t + \mathbf{1}_{\{\tau<\infty\}}\int_\tau^\infty \mathcal{H}_t\left(c_t - I + h_3 L\epsilon\right)\mathrm{d}t\right] \leqslant x, \tag{4.1}$$

where $\mathcal{H}_t$ represents a stochastic discount factor satisfying the follows equation

$$H_t \equiv \zeta_t \mathcal{Z}_t = \exp\left\{-\left(r + \frac{1}{2}\theta^2\right)t - \theta B_t\right\}.$$

Let's start constructing the Lagrangian framework. For any quadruple (c, π, Ξ, τ) in the set $\mathcal{A}(x)$ given by (3.3) and a multiplier $\lambda > 0$, the Lagrangian $\mathcal{L}$ can be defined using the budget constraint given in (4.1) as follows:

$$\begin{aligned}
\mathcal{L} \quad \equiv \quad & \mathbb{E}\left[\int_0^\tau e^{-\beta t}\left(h_1^{\gamma-\gamma^*}\frac{c_t^{1-\gamma}}{1-\gamma} - y_t c_t + y_t(I - h_1\xi\delta)\right)\mathbf{1}_{\{\Xi_t = H_1\}}\mathrm{d}t + \int_0^\tau e^{-\beta t}\left(h_2^{\gamma-\gamma^*}\frac{c_t^{1-\gamma}}{1-\gamma} - y_t c_t + y_t(I - h_2\xi\delta)\right)\mathbf{1}_{\{\Xi_t = H_2\}}\mathrm{d}t\right.\\
& \left. + \mathbf{1}_{\{\tau<\infty\}}e^{-\beta\tau}\int_\tau^\infty e^{-\beta(t-\tau)}\left(\kappa^{\frac{1-\gamma}{\alpha}}h_3^{\gamma-\gamma^*}\frac{c_t^{1-\gamma}}{1-\gamma} - y_t c_t + y_t(I - h_3 L\epsilon)\right)\mathrm{d}t\right] + \lambda x\\
\leqslant \quad & \mathbb{E}\left[\int_0^\tau e^{-\beta t}\left(\tilde{u}_1(y_t)\mathbf{1}_{\{\Xi_t = H_1\}} + \tilde{u}_2(y_t)\mathbf{1}_{\{\Xi_t = H_2\}}\right)\mathrm{d}t + \mathbf{1}_{\{\tau<\infty\}}e^{-\beta\tau}\int_\tau^\infty e^{-\beta(t-\tau)}\tilde{u}_3(y_t)\,\mathrm{d}t\right] + \lambda x,
\end{aligned}$$

where $y_t \equiv \lambda e^{\beta t}H_t$ denotes the dual variable having the differential form of

$$\frac{\mathrm{d}y_t}{y_t} \quad = \quad (\beta - r)\mathrm{d}t - \theta\mathrm{d}B_t, \quad y_0 = \lambda, \tag{4.2}$$

and the convex conjugate utility functions $\tilde{u}$ are deduced as follows:

$$\tilde{u}_i(y_t) \equiv \sup_{c_t>0}\left(h_i^{\gamma-\gamma^*}\frac{c_t^{1-\gamma}}{1-\gamma} - y_t c_t\right) + y_t(I - h_i\xi\delta) = \frac{\gamma}{1-\gamma}h_i^{\frac{\gamma-\gamma^*}{\gamma}}y_t^{-\frac{1-\gamma}{\gamma}} + y_t(I - h_i\xi\delta), \quad \text{for } i \in \{1, 2\}, \tag{4.3}$$

$$\tilde{u}_3(y_t) \equiv \sup_{c_t>0}\left(\kappa^{\frac{1-\gamma}{\alpha}}h_3^{\gamma-\gamma^*}\frac{c_t^{1-\gamma}}{1-\gamma} - y_{1,t}c_t\right) + y_t(I - h_3 L\epsilon) = \frac{\gamma}{1-\gamma}\kappa^{\frac{1-\gamma}{\alpha\gamma}}h_3^{\frac{\gamma-\gamma^*}{\gamma}}y_t^{-\frac{1-\gamma}{\gamma}} + y_t(I - h_3 L\epsilon). \tag{4.4}$$

A direct application of the first-order conditions (FOCs) to (4.3) and (4.4) generates the subsequent optimal consumption

$$c^*(y_t) = \begin{cases} h_i^{\frac{\gamma-\gamma^*}{\gamma}}y_t^{-\frac{1}{\gamma}}, & \text{for } 0 \leqslant t < \tau \text{ and } \Xi_t = H_i,\\ \kappa^{\frac{1-\gamma}{\alpha\gamma}}h_3^{\frac{\gamma-\gamma^*}{\gamma}}y_t^{-\frac{1}{\gamma}}, & \text{for } t \geq \tau \text{ and } \Xi_t = H_3. \end{cases} \tag{4.5}$$

Moreover, during the chonsei period, it is easy to observe that $\tilde{u}_1$ and $\tilde{u}_2$ holds the following relationship

$$\begin{cases} \tilde{u}_2(y_t) \geqslant \tilde{u}_1(y_t), & \text{for } 0 < y_t \leqslant \hat{y},\\ \tilde{u}_2(y_t) < \tilde{u}_1(y_t), & \text{for } y_t > \hat{y}. \end{cases} \tag{4.6}$$

When $\tilde{u}_1(\hat{y}) = \tilde{u}_2(\hat{y})$, it signifies an equilibrium point where the dual utilities linked to both house types are equal, leading to the determination of the following switching boundary $\hat{y}$, i.e.,

$$\hat{y} \equiv \left(\frac{\gamma}{1-\gamma}\frac{h_2^{\frac{\gamma-\gamma^*}{\gamma}} - h_1^{\frac{\gamma-\gamma^*}{\gamma}}}{(h_2 - h_1)\xi\delta}\right)^\gamma > 0. \tag{4.7}$$

Therefore, the relationship in (4.6) says that the optimal chonsei house type during the rental period can be expressed as follows:

$$\Xi(y_t) = \begin{cases} H_2, & \text{for } 0 < y_t \leqslant \hat{y}, \\ H_1, & \text{for } y_t > \hat{y}. \end{cases} \tag{4.8}$$

Through the aforementioned derivation, we ultimately transform the original Problem 1 into the following dual problem. By solving this dual problem, we obtain the optimal consumption and investment strategies for individuals during both the rental and homeownership period.

Problem (Dual problem)**.** *With a positive dual variable λ, we can obtain the dual value function $\phi(\lambda)$ as follows:*

$$\phi(\lambda) \;=\; \sup_{h_3,\tau\in\mathcal{T}} \mathbb{E}\left[\int_0^\tau e^{-\beta t}\left(\tilde{u}_1(y_t)\mathbf{1}_{\{y_t>\hat{y}\}} + \tilde{u}_2(y_t)\mathbf{1}_{\{0<y_t\leqslant\hat{y}\}}\right)\mathrm{d}t + \mathbf{1}_{\{\tau<\infty\}}e^{-\beta\tau}\int_\tau^\infty e^{-\beta(t-\tau)}\tilde{u}_3(y_t)\mathrm{d}t\right]. \tag{4.9}$$

Here, $\mathcal{T}$ encompasses all $\mathcal{F}$-stopping times ranging from 0 to ∞.

Employing the approach delineated in Theorem 8.5 and Corollary 8.7 in the seminal work by Karatzas and Wang (2000), we delve into the relationship of duality, thereby facilitating the derivation of an explicit solution for $V(x)$. This discussion is further elaborated upon in the following remark.

Remark 1. *The relationship between the primal Problem 1 and its dual problem satisfies the following equality*

$$V(x) = \inf_{\lambda>0}\left[\phi(\lambda) + \lambda x\right]. \tag{4.10}$$

Moreover, the implementation of the problem of minimizing (4.10) can yield the unique solution λ^*, complying with the following equation

$$x = -\phi'(\lambda^*). \tag{4.11}$$

In order to reflect the temporal scope of the solution to the function (4.9), we propose the introduction of a binary function $\psi(t, y)$, characterized by its dependence on both time and a dual variable, as outlined below:

$$\psi(t,y) \;\equiv\; \sup_{\tau\in\mathcal{T}} \mathbb{E}^{y_t=y}\left[\int_t^\tau \mathrm{e}^{-\beta s}\left(\tilde{u}_1(y_s)\mathbf{1}_{\{\hat{y}<y_s\}} + \tilde{u}_2(y_s)\mathbf{1}_{\{0<y_s\leqslant\hat{y}\}}\right)\mathrm{d}s + e^{-\beta\tau}\widetilde{U}(y_\tau)\right], \tag{4.12}$$

where

$$\widetilde{U}(y_\tau) \;\equiv\; \int_\tau^\infty e^{-\beta(t-\tau)}\tilde{u}_3(y_t)\mathrm{d}t = \kappa^{\frac{1-\gamma}{\alpha\gamma}} h_3^{\frac{\gamma-\gamma^*}{\gamma}} \frac{\gamma}{(1-\gamma)\,\mathrm{K}} y_\tau^{\frac{\gamma-1}{\gamma}} + \frac{I - h_3 L\epsilon}{r} y_\tau.$$

Remark 2. *Let us establish the subsequent quadratic expression:*

$$f(n) \;\equiv\; \frac{1}{2}\theta^2 n^2 + \left(\beta - r - \frac{1}{2}\theta^2\right)n - \beta = 0.$$

The solutions to the equation $f = 0$ are determined as follows:

$$n_1 = \frac{-\left(\beta - r - \frac{1}{2}\theta^2\right) - \sqrt{\left(\beta - r - \frac{1}{2}\theta^2\right)^2 + 2\theta^2\beta}}{\theta^2} < 0,$$

$$n_2 = \frac{-\left(\beta - r - \frac{1}{2}\theta^2\right) + \sqrt{\left(\beta - r - \frac{1}{2}\theta^2\right)^2 + 2\theta^2\beta}}{\theta^2} > 1.$$

Adhering to the methodology elucidated in Karatzas and Wang (2000) and Li et al. (2024b), we undertake the transformation of the function delineated in (4.12) into a corresponding variational inequality. It is imperative to underscore that the resolution derived from this variational inequality not only provides a solution to the immediate problem at hand but also effectively addresses the optimization task encapsulated by (4.12).

Variational Inequality 1. *Ascertain a free boundary $\bar{y}$, which is equal to y_τ, and a function $\psi(\cdot,\cdot) \in C^1\left((0,\infty)\times\mathbb{R}^+\right) \cap C^2\left((0,\infty)\times(\mathbb{R}^+\backslash\{\bar{y}\})\right)$ which fulfills the subsequent conditions:*

$$\mathcal{L}^*\psi + e^{-\beta t}\left(\tilde{u}_1(y)\mathbf{1}_{\{\hat{y}<y\}} + \tilde{u}_2(y)\mathbf{1}_{\{0<y\leqslant\hat{y}\}}\right) = 0, \quad y > \bar{y}, \tag{4.13}$$

$$\mathcal{L}^*\psi + e^{-\beta t}\left(\tilde{u}_1(y)\mathbf{1}_{\{\hat{y}<y\}} + \tilde{u}_2(y)\mathbf{1}_{\{0<y\leqslant\hat{y}\}}\right) < 0, \quad 0 < y \leqslant \bar{y}, \tag{4.14}$$

$$\psi(t,y) > e^{-\beta t}\widetilde{U}(y), \quad y > \bar{y}, \tag{4.15}$$

$$\psi(t,y) = e^{-\beta t}\widetilde{U}(y), \quad 0 < y \leqslant \bar{y}, \tag{4.16}$$

where $\mathcal{L}^*$ can be defined by

$$\mathcal{L}^* \equiv \frac{\partial}{\partial t} + (\beta - r)y\frac{\partial}{\partial y} + \frac{1}{2}\theta^2 y^2 \frac{\partial^2}{\partial y^2}.$$

Proposition 1. *For $\psi(t,y)$ mentioned above, let $\psi(t,y) \equiv \mathrm{e}^{-\beta t}v(y)$. And the function $v(y)$ is expressed by*

$$v(y) = \begin{cases} By^{n_1} + h_1^{\frac{\gamma-\gamma^*}{\gamma}}\frac{\gamma}{(1-\gamma)K}y^{\frac{\gamma-1}{\gamma}} + \frac{I-h_1\xi\delta}{r}y, & \text{for } y > \hat{y}, \\ A_1 y^{n_2} + A_2 y^{n_1} + h_2^{\frac{\gamma-\gamma^*}{\gamma}}\frac{\gamma}{(1-\gamma)K}y^{\frac{\gamma-1}{\gamma}} + \frac{I-h_2\xi\delta}{r}y, & \text{for } \bar{y} < y \leqslant \hat{y}, \\ \kappa^{\frac{1-\gamma}{\alpha\gamma}} h_3^{\frac{\gamma-\gamma^*}{\gamma}}\frac{\gamma}{(1-\gamma)K}y^{\frac{\gamma-1}{\gamma}} + \frac{I-h_3L\epsilon}{r}y, & \text{for } 0 < y \leqslant \bar{y}, \end{cases}$$

where A_1, A_2, and B are constant coefficients, described by

$$A_1 = \frac{\frac{\gamma n_1+1-\gamma}{\gamma K} + \frac{1-n_1}{r}}{(n_2-n_1)\,\hat{y}^{n_2-1}}\,(h_2-h_1)\,\xi\delta, \tag{4.17}$$

$$A_2 = \frac{\frac{\gamma n_2+1-\gamma}{K(1-\gamma)}\bar{y}^{-\frac{1}{\gamma}}\left(\kappa^{\frac{1-\gamma}{\alpha\gamma}}h_3^{\frac{\gamma-\gamma^*}{\gamma}} - h_2^{\frac{\gamma-\gamma^*}{\gamma}}\right) + \left(\frac{h_2\xi\delta - h_3L\epsilon}{r} - (1-L)h_3\right)(n_2-1)}{(n_2-n_1)\bar{y}^{n_1-1}}, \tag{4.18}$$

$$\begin{aligned} B = {} & \frac{\frac{\gamma n_1+1-\gamma}{\gamma K} + \frac{1-n_1}{r}}{(n_2-n_1)\,\hat{y}^{n_1-1}}\,(h_2-h_1)\,\xi\delta + \frac{\frac{\gamma n_2+1-\gamma}{K(1-\gamma)}\bar{y}^{-\frac{1}{\gamma}}\left(\kappa^{\frac{1-\gamma}{\alpha\gamma}}h_3^{\frac{\gamma-\gamma^*}{\gamma}} - h_2^{\frac{\gamma-\gamma^*}{\gamma}}\right) + \left(\frac{h_2\xi\delta - h_3L\epsilon}{r} - (1-L)h_3\right)(n_2-1)}{(n_2-n_1)\bar{y}^{n_1-1}} \\ & + \frac{\gamma}{(1-\gamma)\,K}\hat{y}^{\frac{\gamma-1}{\gamma}-n_1}\left(h_2^{\frac{\gamma-\gamma^*}{\gamma}} - h_1^{\frac{\gamma-\gamma^*}{\gamma}}\right) + \frac{(h_1-h_2)\xi\delta}{r}\hat{y}^{1-n_1}. \end{aligned} \tag{4.19}$$

The free boundary $\bar{y}$ conforms to the following equality, i.e.,

$$\begin{aligned} & -\frac{\gamma n_1 + 1 - \gamma}{(1-\gamma)K}\bar{y}^{-\frac{1}{\gamma}}\left[\kappa^{\frac{1-\gamma}{\alpha\gamma}}\left(\kappa^{-\frac{1-\gamma}{\alpha\gamma}}\frac{\gamma(1-\gamma)K}{\gamma-\gamma^*}(1-L)\,\bar{y}^{\frac{1}{\gamma}}\right)^{-\frac{\gamma-\gamma^*}{\gamma^*}} - h_2^{\frac{\gamma-\gamma^*}{\gamma}}\right] \\ = {} & \left[\frac{h_2\xi\delta}{r} - \left(\frac{\epsilon - r}{r}L + 1\right)\left(\kappa^{-\frac{1-\gamma}{\alpha\gamma}}\frac{\gamma(1-\gamma)K}{\gamma-\gamma^*}(1-L)\,\bar{y}^{\frac{1}{\gamma}}\right)^{-\frac{\gamma}{\gamma^*}}\right](n_1-1) + \frac{\frac{\gamma n_1+1-\gamma}{\gamma K} + \frac{1-n_1}{r}}{\hat{y}^{n_2-1}}\,(h_2-h_1)\,\xi\delta\bar{y}^{n_2-1}. \end{aligned} \tag{4.20}$$

And the optimal magnitude of home purchase is determined by

$$h_3 \quad = \quad \left[\kappa^{-\frac{1-\gamma}{\alpha\gamma}} \frac{\gamma(1-\gamma)K}{\gamma-\gamma^*} (1-L)\,\bar{y}^{\frac{1}{\gamma}}\right]^{-\frac{\gamma}{\gamma^*}}.$$

Consequently, $\psi(t, y) = \mathrm{e}^{-\beta t} v(y)$ is a solution to Variational Inequality 1.

Proof. See Appendix A. □

4.2. Optimal Solutions

It is apparent that the equalities $\phi(\lambda) = \psi(0, \lambda) = v(\lambda)$ hold true. Utilizing Proposition 1 in conjunction with the dual relationship elucidated in Remark 1 generates the solution of the value function $V(x)$ straightforwardly as follows:

Theorem 1. *The value function for the optimal stopping problem stated in (3.3) can be delineated as*

$$V(x) = \begin{cases} B(1-n_1)\left(\lambda_1^*\right)^{n_1} + h_1^{\frac{\gamma-\gamma^*}{\gamma}} \frac{1}{(1-\gamma)K}\left(\lambda_1^*\right)^{-\frac{1-\gamma}{\gamma}}, & \text{for } \lambda_1^* > \hat{y} \text{ with } t < \tau^*, \\ A_1(1-n_2)\left(\lambda_2^*\right)^{n_2} + A_2(1-n_1)\left(\lambda_2^*\right)^{n_1} + h_2^{\frac{\gamma-\gamma^*}{\gamma}} \frac{1}{(1-\gamma)K}\left(\lambda_2^*\right)^{-\frac{1-\gamma}{\gamma}}, & \text{for } \bar{y} < \lambda_2^* \leqslant \hat{y} \text{ with } t < \tau^*, \\ \frac{\kappa^{\frac{1-\gamma}{\alpha}} h_3^{\gamma-\gamma^*}}{(1-\gamma)K^\gamma}\left(x + \frac{I - h_3 L\epsilon}{r}\right)^{1-\gamma}, & \text{for } \tau^* \leqslant t, \end{cases}$$

where λ_1^*, λ_2^* and λ_3^* are established through the subsequent algebraic equations

$$x = -n_1 B\,(\lambda_1^*)^{n_1-1} + h_1^{\frac{\gamma-\gamma^*}{\gamma}} \frac{1}{K}\,(\lambda_1^*)^{-\frac{1}{\gamma}} - \frac{I - h_1\xi\delta}{r},$$

$$x = -n_2 A_1\,(\lambda_2^*)^{n_2-1} - n_1 A_2\,(\lambda_2^*)^{n_1-1} + h_2^{\frac{\gamma-\gamma^*}{\gamma}} \frac{1}{K}\,(\lambda_2^*)^{-\frac{1}{\gamma}} - \frac{I - h_2\xi\delta}{r},$$

and

$$x = \kappa^{\frac{1-\gamma}{\alpha\gamma}} h_3^{\frac{\gamma-\gamma^*}{\gamma}} \frac{1}{K}\left(\lambda_3^*\right)^{-\frac{1}{\gamma}} - \frac{I - h_3 L\epsilon}{r},$$

respectively. The optimal time for homebuying, denoted as τ^*, can be described by

$$\tau^* \quad = \quad \inf\{t \geq 0 \mid 0 < y_t^* \leqslant \bar{y}\}.$$

Considering that $y_t^{\lambda_p^*}$ represents the solution to the stochastic differential equation (4.2) under the starting point λ_p^*, where p ranges within the set $\{1, 2, 3\}$, the optimal wealth level X_t^* can be directly computed using (4.11) as delineated in Remark 1. This procedure yields the following:

Remark 3. *The expression of the optimal solution for the process X_t^* is formulated as follows:*

$$X_t^* \quad = \quad \begin{cases} -n_1 B\left(y_t^{\lambda_1^*}\right)^{n_1-1} + h_1^{\frac{\gamma-\gamma^*}{\gamma}} \frac{1}{K}\left(y_t^{\lambda_1^*}\right)^{-\frac{1}{\gamma}} - \frac{I-h_1\xi\delta}{r}, & \text{for } y_t^{\lambda_1^*} > \hat{y} \text{ with } t < \tau^*, \\ -n_2 A_1\left(y_t^{\lambda_2^*}\right)^{n_2-1} - n_1 A_2\left(y_t^{\lambda_2^*}\right)^{n_1-1} + h_2^{\frac{\gamma-\gamma^*}{\gamma}} \frac{1}{K}\left(y_t^{\lambda_2^*}\right)^{-\frac{1}{\gamma}} - \frac{I-h_2\xi\delta}{r}, & \text{for } \bar{y} < y_t^{\lambda_2^*} \leqslant \hat{y} \text{ with } t < \tau^*, \\ \kappa^{\frac{1-\gamma}{\alpha\gamma}} h_3^{\frac{\gamma-\gamma^*}{\gamma}} \frac{1}{K}\left(y_t^{\lambda_3^*}\right)^{-\frac{1}{\gamma}} - \frac{I-h_3L\epsilon}{r}, & \text{for } \tau^* \leqslant t. \end{cases} \tag{4.21}$$

In addition, the wealth threshold for chonsei-switching is determined by

$$\hat{x} = -n_2 A_1 \hat{y}^{n_2-1} - n_1 A_2 \hat{y}^{n_1-1} + h_2^{\frac{\gamma-\gamma^*}{\gamma}} \frac{1}{K} \hat{y}^{-\frac{1}{\gamma}} - \frac{I - h_2 \xi \delta}{r},$$

and the determination of the wealth threshold for homebuying is accomplished by

$$\bar{x} = \kappa^{\frac{1-\gamma}{\alpha\gamma}} h_3^{\frac{\gamma-\gamma^*}{\gamma}} \frac{1}{K} \bar{y}^{-\frac{1}{\gamma}} - \frac{I - h_3 L \epsilon}{r}.$$

Prior to owning a house, when $y_t^{\lambda_1^*} > \hat{y}$, the presence of the first term in (4.21) is attributed to the choice to transition the chonsei house type. Conversely, if $\bar{y} < y_t^{\lambda_2^*} \leqslant \hat{y}$ before homeownership, the emergence of the first term in (4.21) is due to the upper boundary constraint on $y_t^{\lambda_2^*}$, while the second term arises from the decision to acquire a house. Subsequent to the home purchase decision, the wealth solution adopts a simplified form, as the individual focuses solely on determining consumption and risky asset investment strategies.

Theorem 2. *The optimal trajectories of the house type process $(\Xi_t^*)_{t\geq 0}$ and the consumption process $(c_t^*)_{t\geq 0}$ are established by*

$$\Xi_t^* = \begin{cases} H_1, & \text{for } y_t^{\lambda_1^*} > \hat{y} \text{ with } t < \tau^*, \\ H_2, & \text{for } \bar{y} < y_t^{\lambda_2^*} \leqslant \hat{y} \text{ with } t < \tau^*, \\ H_3, & \text{for } \tau^* \leqslant t, \end{cases} \quad c_t^* = \begin{cases} h_1^{\frac{\gamma-\gamma^*}{\gamma}} \left(y_t^{\lambda_1^*}\right)^{-\frac{1}{\gamma}}, & \text{for } y_t^{\lambda_1^*} > \hat{y} \text{ with } t < \tau^*, \\ h_2^{\frac{\gamma-\gamma^*}{\gamma}} \left(y_t^{\lambda_2^*}\right)^{-\frac{1}{\gamma}}, & \text{for } \bar{y} < y_t^{\lambda_2^*} \leqslant \hat{y} \text{ with } t < \tau^*, \\ \kappa^{\frac{1-\gamma}{\alpha\gamma}} h_3^{\frac{\gamma-\gamma^*}{\gamma}} \left(y_t^{\lambda_3^*}\right)^{-\frac{1}{\gamma}}, & \text{for } \tau^* \leqslant t, \end{cases}$$

and the optimal portfolio strategy $(\pi_t^*)_{t\geq 0}$ is provided by

$$\pi_t^* = \begin{cases} \frac{\theta}{\sigma} F_1\left(y_t^{\lambda_1^*}\right), & \text{for } y_t^{\lambda_1^*} > \hat{y} \text{ with } t < \tau, \\ \frac{\theta}{\sigma} F_2\left(y_t^{\lambda_2^*}\right), & \text{for } \bar{y} < y_t^{\lambda_2^*} < \hat{y} \text{ with } t < \tau, \\ \frac{\theta}{\sigma} F_3\left(y_t^{\lambda_3^*}\right), & \text{for } \tau \leqslant t, \end{cases}$$

where

$$\begin{aligned} F_1\left(y_t^{\lambda_1^*}\right) &= n_1(n_1-1) B \left(y_t^{\lambda_1^*}\right)^{n_1-1} + h_1^{\frac{\gamma-\gamma^*}{\gamma}} \frac{1}{\gamma K} \left(y_t^{\lambda_1^*}\right)^{-\frac{1}{\gamma}}, \\ F_2\left(y_t^{\lambda_2^*}\right) &= n_2(n_2-1) A_1 \left(y_t^{\lambda_2^*}\right)^{n_2-1} + n_1(n_1-1) A_2 \left(y_t^{\lambda_2^*}\right)^{n_1-1} + h_2^{\frac{\gamma-\gamma^*}{\gamma}} \frac{1}{\gamma K} \left(y_t^{\lambda_2^*}\right)^{-\frac{1}{\gamma}}, \\ F_3\left(y_t^{\lambda_3^*}\right) &= \kappa^{\frac{1-\gamma}{\alpha\gamma}} h_3^{\frac{\gamma-\gamma^*}{\gamma}} \frac{1}{\gamma K} \left(y_t^{\lambda_3^*}\right)^{-\frac{1}{\gamma}}. \end{aligned}$$

In addition, the values of $y_t^{\lambda_1^*}$, $y_t^{\lambda_2^*}$, and $y_t^{\lambda_3^*}$ are deduced by (4.21).

Proof. See Appendix B. □

Based on the analysis of housing type, during the chonsei rental phase, individuals tend to opt for smaller chonsei residences initially, later transitioning to larger ones. This highlights a cost-oriented viewpoint wherein individuals residing in smaller chonsei dwellings incur a lower housing expense but enjoy a higher income $(I - h_1 \xi \delta)$ compared to when residing in larger accommodations.

5. Economic Implications

In this section, we conduct various comparative statics using the closed-form solution derived in () - (). Our main focus is on the housing-related decision-making of this agent, specifically regarding the choice of housing size and the form of residential service acquisition, namely the choice between chonsei and purchasing a house. While consumption and investment choices, as described in the preceding model, are also crucial variables in decision-making, they have been sufficiently analyzed in various previous studies. Therefore, we will briefly discuss about them.

However, it is very difficult to do the comparative statics analytically using the solution in equations () to (). Instead, we do through numerical demonstration and the baseline numerical values for parameters required for this are as follows:

$$\kappa = 1.5;\ L = 0.6;\ \delta = 0.025;\ \xi = 0.6;\ \epsilon = 0.03;\ r = 0.04;\ \beta = 0.05;$$
$$\gamma = 2;\ \mu = 0.09;\ \sigma = 0.2;\ I = 1;\ h_1 = 15;\ h_2 = 20;\ \alpha = 0.5. \tag{5.1}$$

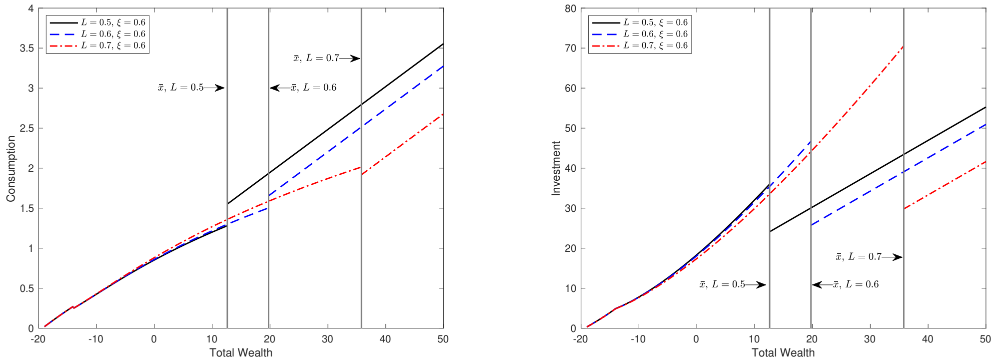
The values in (5.1) are set by referencing various papers such as Li and Ahn (2022), Li et al. (2024a), Ahn and Ryu (2024), and Jorda et al. (2019). Among these, L and ξ, representing the LTV ratio and the chonsei-to-price ratio, are respectively set to reflect the ratios in the market. Various levels of values of them will be applied in the subsequent analysis. Additionally, we set $\delta < \epsilon < r$, so that (i) the interest rate on housing loans is lowest due to government assistance for rental deposits through the chonsei guarantee, and (ii) the mortgage interest rate is lower than the interest rate on loans without a collateral.

5.1. LTV and Chonsei-to-Price Ratios

This subsection examines the impact of changes in the LTV ratio and the chonsei-to-price ratio on housing choices. Figure 2 shows the optimal consumption and investment according to changes in the LTV ratio, particularly highlighting changes in the optimal timing for purchasing a house.

The most notable feature is that as the LTV ratio increases, households that are renting a house through the chonsei system tend to delay their home purchase, which is similar to the findings of Li et al. (2024a). As the LTV increases, the amount that these households need to pay when purchasing a house decreases. Using the notation of the model, this means that the amount that needs to be transferred from financial wealth to housing wealth decreases. Therefore, households with relatively greater liquidity purchase larger homes as the LTV increases. This is well illustrated in the following figure.

However, as the LTV ratio increases and households aim to purchase larger homes, they will have to manage relatively larger mortgage loans. This increases the interest they need to pay over their lifetime. To handle this additional interest burden, these households will seek to secure more financial wealth before purchasing a house. Therefore, as the LTV ratio increases, these households delay their home purchase while simultaneously aiming to buy a larger home.



(a) Consumption (b) Investment

Figure 2: Optimal Policies w.r.t. L

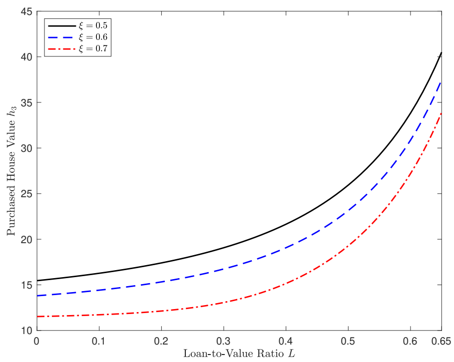
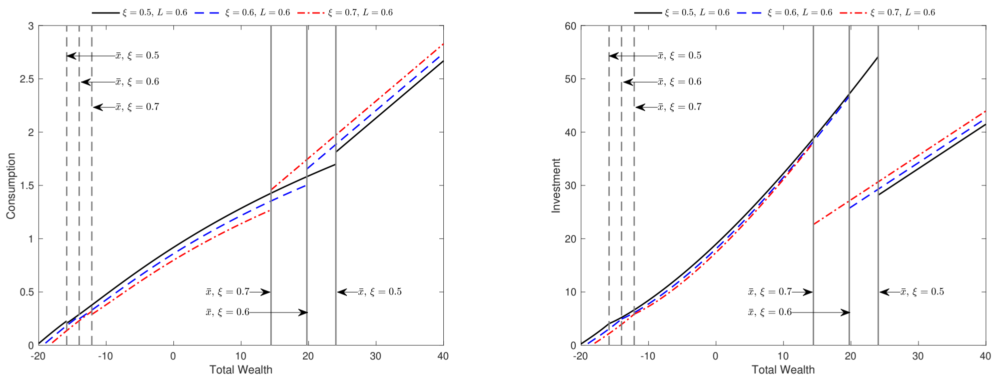


Figure 3: Optimal Purchasing of a House w.r.t. L

On the other hand, the impact of changes in the chonsei-to-price ratio on the timing of home purchases is well illustrated in Figure 4.

As shown clearly in Figure 4, when the chonsei-to-price ratio increases, these households advance the timing of their home purchase. First, it is important to note that changes in the chonsei-to-price ratio have a smaller impact on the financial wealth of these households compared to changes in the LTV ratio. In the case of purchasing a house, households need to cover the difference between the price of the house and the amount financed through a mortgage loan. However, in the case of chonsei, the process concludes by borrowing the necessary deposit amount through a chonsei deposit loan. In other words, changes in the chonsei-to-price ratio have little impact in terms of liquidity. However, if the chonsei-to-price ratio increases, resulting in a larger loan, the interest costs naturally increase. Therefore, changes in the chonsei-to-price ratio affect these households' decision-making purely from a cost perspective.

Additionally, as shown in Figure 3, when the chonsei-to-price ratio increases, the size of the purchased home



(a) Consumption

(b) Investment

Figure 4: Sensitivity w.r.t. ξ

decreases. This is because, although the household advances the purchase of house to avoid the much interest on a large chonsei loan, the available funds for purchasing are reduced, leading them to buy relatively smaller houses.

Another point to consider is that an increase in the chonsei-to-price ratio could (albeit differentially) boost housing demand. In Korea, it is often said that chonsei prices support housing prices, and this phenomenon can be observed in Figure 4. As the chonsei-to-price ratio increases, households facing higher chonsei living costs may switch to purchasing earlier. This creates additional demand for home purchases, which can, in turn, exert upward pressure on housing prices.

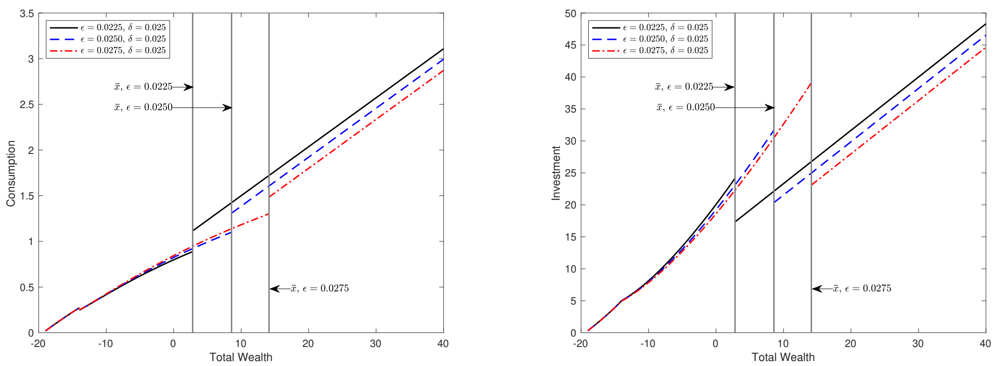
Compared to Figure 2, the most notable point in Figure 4 is that changes in the chonsei-to-price ratio also affect the size of rented house chosen under the chonsei system. In Figure 2, there was little difference in the choice of rented house by chonsei. However, as in Figure 4, as the chonsei-to-price ratio increases, households tend to stay longer in relatively smaller rental houses to manage the increased costs. Therefore, when the chonsei-to-price ratio increases, households tend to live in smaller houses by chonsei and purchase houses relatively earlier. This suggests a potential decrease in not only the chonsei demand for larger houses but also the demand for purchasing larger houses.

Optimal consumption and investment in models similar to this one, involving an optimal stopping problem, have been deeply analyzed in various prior studies, so we will provide a brief explanation. Looking again at Figure 2, it is evident that as the time to purchase a house approaches, households reduce consumption and increase investment. This is to secure funds for the home purchase as quickly as possible. Specifically, Figure 2 shows that the larger the LTV ratio, the more consumption and investment decrease after the home purchase. This is due to increased interest costs from the larger loan amount. However, consumption and investment follow different decision-making processes. For investment, the decrease can be simply attributed to the increased lifetime costs due to higher interest costs. Consumption, however, behaves differently. Considering the results in Figure 3, as the LTV ratio increases, households end up living in larger houses. The reduction in utility from decreased consumption is offset by the increased utility from living in a larger house. This consumption pattern is also observed in Figures 4 through 6, later. If households purchase a house early, moving out of chonsei, they might live in relatively smaller houses due to a lack of financial wealth. However, the reduced utility from living in a smaller house can be sufficiently offset by the increased consumption due to lower housing costs after the purchase.

This indicates that major variables in the housing market significantly impact not only housing-related decisions but also consumption and investment. Given that housing assets or chonsei deposits constitute a substantial portion of most households' assets, this is a natural result. In particular, please notice that the LTV ratio is usually set by the authority, but the chonsei-to-price ratio is determined in the housing market. Policies that expand the LTV ratio may boost housing demand but could also lead to a decrease in consumption and investment due to increased housing costs. Households, taking full advantage of the opportunity to secure larger mortgage loans, tend to purchase larger homes, leading to higher interest costs. This results in a decrease in disposable income after paying interest, naturally reducing consumption and investment. Consequently, policies aimed at stimulating the housing market might inadvertently cause a contraction in the overall consumption-investment market, leading to unintended negative effects.

5.2. *Interest Rates on the Mortgage Loans and Chonsei Loans*

In this subsection, we analyze the impact of changes in variables related to the financial costs of obtaining housing services on housing choices. Figure 5 illustrates the impact of changes in mortgage loan interest rates on the timing of home purchases.

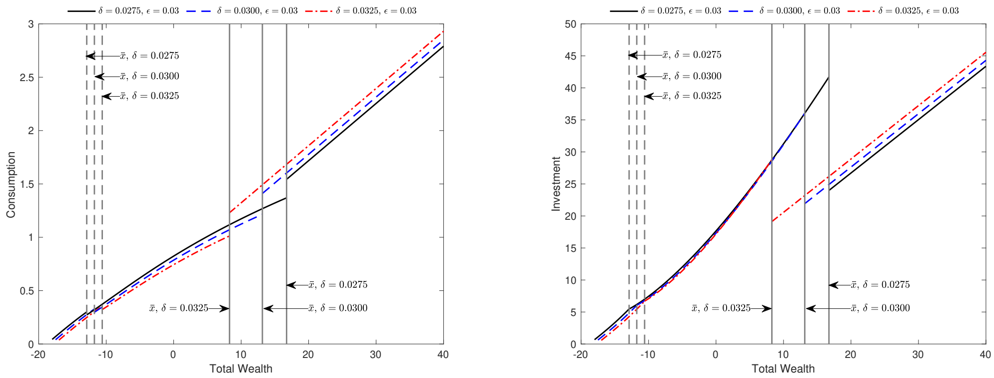


(a) Consumption (b) Investment

Figure 5: Sensitivity w.r.t. ϵ

As shown in the figure, the higher the interest rate on mortgage loans, the more households delay their home purchase. This is because the interest rate acts as a lifetime cost, requiring households to secure sufficient financial wealth to cover it. This argument is similar to the one regarding the impact of the LTV ratio in Figure 2.

Figure 6 shows the relationship between changes in chonsei deposit loan interest rates and the timing of housing changes. When the interest rate on chonsei deposit loans increases, it directly leads to higher costs for the households under chonsei. As a result, households tend to purchase homes earlier and, if they must continue residing in a house by chonsei, they prefer to stay in smaller ones for a longer period.



(a) Consumption (b) Investment

Figure 6: Sensitivity w.r.t. δ

Here, it is worth noting that while the interest rate on mortgage loans is determined in the financial market, the interest rate on chonsei deposit loans is policy-driven through government guarantees. Considering this, the following implications can be drawn: The government may keep the interest rate on chonsei deposit loans significantly lower than other market interest rates as a policy measure to assist low-income individuals in housing stability. While this would directly reduce the housing costs for households living in chonsei rental housing, it could conversely decrease the willingness of households with potential intentions to purchase a house, leading to a decrease in demand in the housing market.

6. Conclusion

In this study, we addressed the optimal housing selection model for households in a rigorous continuous-time optimal portfolio selection framework, covering both chonsei and home purchase decisions. Specifically, we depicted home purchase as an optimal stopping problem and by allowing chonsei switching, modeled household choices with various options close to reality . Subsequently, we used the martingale method to derive closed-form solutions and conducted various comparative statics.

The main findings of this study are as follows: Analyzing the influence of LTV and chonsei-to-price ratio, which serve as loan limits, it was observed that as the LTV ratio increases and the chonsei-to-price ratio decreases, households tend to purchase houses earlier and opt for smaller houses. Additionally, an increase in the LTV ratio was associated with a decrease in consumption and investment following home purchase. Considering that LTV is generally determined by government policy, it was shown that government housing-related policies may have unintended negative side effects on the consumption-investment market. Examining the impact of two interest rates representing housing costs, mortgage interest rate, and chonsei loan interest rate, it was found that households naturally gravitate towards relatively cheaper options. Particularly noteworthy is the chonsei loan interest rate, which being policy-driven, demonstrated that government policies aimed at assisting chonsei renters may have unexpected side effects on the housing market.

The limitation of this study lies in its focus solely on tenants' models, which imposes constraints on modeling the vast housing and financial markets. Therefore, if we were able to analyze a general equilibrium model that includes landlords, financial institutions, and even government guarantee-related decisions, we could provide a comprehensive perspective on these markets and offer policy recommendations.

Appendix A. Detailed Proof of Proposition 1

We concentrate on solving the equalities (4.13) and (4.16) within Variational Inequality 1. We elucidate that

$$\psi(t,y) \equiv \begin{cases} \psi_1(t,y), & \text{for } y > \hat{y}, \\ \psi_2(t,y), & \text{for } \bar{y} < y \leqslant \hat{y}, \\ \psi_3(t,y), & \text{for } 0 < y \leqslant \bar{y}, \end{cases}$$

which are the solutions to the partial differential equations (PDE) stated in Variational Inequality 1. Then, we have the following PDE system

$$\begin{cases} \frac{\partial \psi_1}{\partial t} + (\beta - r)y\frac{\partial \psi_1}{\partial y} + \frac{1}{2}\left(\theta_1^2 + \theta_2^2\right)y^2\frac{\partial^2 \psi_1}{\partial y^2} + \mathrm{e}^{-\beta t}\left(h_1^{\frac{\gamma-\gamma^*}{\gamma}}\frac{\gamma}{1-\gamma}y^{\frac{\gamma-1}{\gamma}} + (I - h_1\xi\delta)y\right) = 0, & \text{for } y > \hat{y}, \\ \frac{\partial \psi_2}{\partial t} + (\beta - r)y\frac{\partial \psi_2}{\partial y} + \frac{1}{2}\left(\theta_1^2 + \theta_2^2\right)y^2\frac{\partial^2 \psi_2}{\partial y^2} + \mathrm{e}^{-\beta t}\left(h_2^{\frac{\gamma-\gamma^*}{\gamma}}\frac{\gamma}{1-\gamma}y^{\frac{\gamma-1}{\gamma}} + (I - h_2\xi\delta)y\right) = 0, & \text{for } \bar{y} < y \leqslant \hat{y}, \\ \frac{\partial \psi_3}{\partial t} + (\beta - r)y\frac{\partial \psi_3}{\partial y} + \frac{1}{2}\left(\theta_1^2 + \theta_2^2\right)y^2\frac{\partial^2 \psi_3}{\partial y^2} + \mathrm{e}^{-\beta t}\left(\kappa^{\frac{1-\gamma}{\alpha\gamma}}h_3^{\frac{\gamma-\gamma^*}{\gamma}}\frac{\gamma}{1-\gamma}y^{\frac{\gamma-1}{\gamma}} + (I - h_3L\epsilon)y\right) = 0, & \text{for } 0 < y < \bar{y}. \end{cases} \quad (A.1)$$

We explore a test solution characterized by $\psi(t,y) = e^{-\beta t}v(y)$ and insert it into (A.1). Subsequently, we derive the ensuing set of ordinary differential equations (ODE)

$$\begin{cases} \frac{1}{2}\theta^2y^2v''(y) + (\beta - r)yv'(y) - \beta v(y) + h_1^{\frac{\gamma-\gamma^*}{\gamma}}\frac{\gamma}{1-\gamma}y^{\frac{\gamma-1}{\gamma}} + (I - h_1\xi\delta)y = 0, & \text{for } y > \hat{y}, \\ \frac{1}{2}\theta^2y^2v''(y) + (\beta - r)yv'(y) - \beta v(y) + h_2^{\frac{\gamma-\gamma^*}{\gamma}}\frac{\gamma}{1-\gamma}y^{\frac{\gamma-1}{\gamma}} + (I - h_2\xi\delta)y = 0, & \text{for } \bar{y} < y \leqslant \hat{y}, \\ \frac{1}{2}\theta^2y^2v''(y) + (\beta - r)yv'(y) - \beta v(y) + \kappa^{\frac{1-\gamma}{\alpha\gamma}}h_3^{\frac{\gamma-\gamma^*}{\gamma}}\frac{\gamma}{1-\gamma}y^{\frac{\gamma-1}{\gamma}} + (I - h_3L\epsilon)y = 0, & \text{for } 0 < y \leqslant \bar{y}. \end{cases} \quad (A.2)$$

The resolution of ODE system (A.2) yields

$$v(y) = \begin{cases} By^{n_1} + h_1^{\frac{\gamma-\gamma^*}{\gamma}}\frac{\gamma}{(1-\gamma)K}y^{\frac{\gamma-1}{\gamma}} + \frac{I-h_1\xi\delta}{r}y, & \text{for } y > \hat{y}, \\ A_1y^{n_2} + A_2y^{n_1} + h_2^{\frac{\gamma-\gamma^*}{\gamma}}\frac{\gamma}{(1-\gamma)K}y^{\frac{\gamma-1}{\gamma}} + \frac{I-h_2\xi\delta}{r}y, & \text{for } \bar{y} < y \leqslant \hat{y}, \\ \kappa^{\frac{1-\gamma}{\alpha\gamma}}h_3^{\frac{\gamma-\gamma^*}{\gamma}}\frac{\gamma}{(1-\gamma)K}y^{\frac{\gamma-1}{\gamma}} + \frac{I-h_3L\epsilon}{r}y, & \text{for } 0 < y \leqslant \bar{y}, \end{cases} \quad (A.3)$$

where $n_1 < 0$ and $n_2 > 1$ represent the roots of $f = 0$ as described in Remark 2.

It is worth noting that the individual reallocates a portion of financial wealth to house value. This implies that the total wealth level at time τ^- is composed of the financial wealth and house value at time τ^+, i.e.,

$$x_{\tau^-} = x_{\tau^+} + (1-L)h_3. \tag{A.4}$$

In addition, at the time of purchasing, indicated by τ, the following equality holds

$$V(x_{\tau^-}) = \sup_{h_3} \hat{V}(x_{\tau^+}). \tag{A.5}$$

Deriving the FOCs with respect to h_3 for each side of equality (A.5) and utilizing Remark 1, lead to the generation of the following terms:

$$\begin{aligned}
\frac{\partial V(x_{\tau^+} + (1-L)h_3)}{\partial h_3} &= V'(x_{\tau^+} + (1-L)h_3)(1-L) \\
&= \bar{y}^+(1-L), \qquad \text{(A.6)} \\
\frac{\partial \hat{V}(x_{\tau^+})}{\partial h_3} &= \frac{\partial \left(v(\bar{y}^-) + \bar{y}^- x_{\tau^+}\right)}{\partial h_3} \\
&= \kappa^{\frac{1-\gamma}{\alpha\gamma}} \frac{\gamma-\gamma^*}{\gamma(1-\gamma)K} h_3^{-\frac{\gamma^*}{\gamma}} (\bar{y}^-)^{\frac{\gamma-1}{\gamma}}. \qquad \text{(A.7)}
\end{aligned}$$

Therefore, setting (A.6) equal to (A.7) provides the optimal solution for the size of the house to purchase, represented by h_3, namely:

$$h_3 = \left[\kappa^{-\frac{1-\gamma}{\alpha\gamma}} \frac{\gamma(1-\gamma)K}{\gamma-\gamma^*} (1-L)\, \bar{y}^{\frac{1}{\gamma}}\right]^{-\frac{\gamma}{\gamma^*}}.$$

Based on the value-matching and smooth-pasting conditions at $\hat{y}$ and $\bar{y}$ respectively, the following equalities arise:

$$\begin{aligned}
v(\hat{y}^-) &= v(\hat{y}^+), && \text{(A.8)} \\
v'(\hat{y}^-) &= v'(\hat{y}^+), && \text{(A.9)} \\
v(\bar{y}^-) + \bar{y}^- x_{\tau^+} &= v(\bar{y}^+) + \bar{y}^+ x_{\tau^-}, && \text{(A.10)} \\
v'(\bar{y}^-) + x_{\tau^+} &= v'(\bar{y}^+) + x_{\tau^-}. && \text{(A.11)}
\end{aligned}$$

Subsequently, by substituting (A.3) and (A.4) into (A.8), (A.9), (A.10), and (A.11), we can express them as follows:

$$\begin{aligned}
A_1\hat{y}^{n_2} + A_2\hat{y}^{n_1} + h_2^{\frac{\gamma-\gamma^*}{\gamma}} \frac{\gamma}{(1-\gamma)K}\hat{y}^{\frac{\gamma-1}{\gamma}} + \frac{I-h_2\xi\delta}{r}\hat{y} &= B\hat{y}^{n_1} + h_1^{\frac{\gamma-\gamma^*}{\gamma}} \frac{\gamma}{(1-\gamma)K}\hat{y}^{\frac{\gamma-1}{\gamma}} + \frac{I-h_1\xi\delta}{r}\hat{y}, \qquad \text{(A.12)} \\
n_2A_1\hat{y}^{n_2-1} + n_1A_2\hat{y}^{n_1-1} - h_2^{\frac{\gamma-\gamma^*}{\gamma}} \frac{1}{K}\hat{y}^{-\frac{1}{\gamma}} + \frac{I-h_2\xi\delta}{r} &= n_1B\hat{y}^{n_1-1} - h_1^{\frac{\gamma-\gamma^*}{\gamma}} \frac{1}{K}\hat{y}^{-\frac{1}{\gamma}} + \frac{I-h_1\xi\delta}{r}, \qquad \text{(A.13)} \\
\kappa^{\frac{1-\gamma}{\alpha\gamma}} h_3^{\frac{\gamma-\gamma^*}{\gamma}} \frac{\gamma}{(1-\gamma)K}\bar{y}^{\frac{\gamma-1}{\gamma}} + \frac{I-h_3L\epsilon}{r}\bar{y} &= A_1\bar{y}^{n_2} + A_2\bar{y}^{n_1} \\
&\quad + h_2^{\frac{\gamma-\gamma^*}{\gamma}} \frac{\gamma}{(1-\gamma)K}\bar{y}^{\frac{\gamma-1}{\gamma}} + \frac{I-h_2\xi\delta}{r}\bar{y} + \bar{y}(1-L)h_3, \qquad \text{(A.14)} \\
-\kappa^{\frac{1-\gamma}{\alpha\gamma}} h_3^{\frac{\gamma-\gamma^*}{\gamma}} \frac{1}{K}\bar{y}^{-\frac{1}{\gamma}} + \frac{I-h_3L\epsilon}{r} &= n_2A_1\bar{y}^{n_2-1} + n_1A_2\bar{y}^{n_1-1} \\
&\quad - h_2^{\frac{\gamma-\gamma^*}{\gamma}} \frac{1}{K}\bar{y}^{-\frac{1}{\gamma}} + \frac{I-h_2\xi\delta}{r} + (1-L)h_3. \qquad \text{(A.15)}
\end{aligned}$$

From equalities (4.7), (A.12), and (A.13), we derive A_1 as given in (4.17). Similarly, by employing equations (A.14) and (A.15), we obtain A_2 as expressed in (4.18). Substituting (4.17) and (4.18) into (A.14) results in an algebraic equation, and the free boundary $\bar{y}$ corresponds to the root of the equation as outlined in (4.20). Upon substituting (4.7), (4.17), and (4.18) into (A.12), we determine the coefficient B as indicated in (4.19). Thus, $\psi(t,y) = \mathrm{e}^{-\beta t}v(y)$ stands as the solution to equalities (4.13) and (4.16).

Given the well-defined nature of the model, $\psi(t,y) = \mathrm{e}^{-\beta t}v(y)$ meets the criteria outlined in inequalities (4.14) and (4.15) within Variational Inequality 1. Employing similar methods as those detailed in Li et al. (2024b), we can establish the validity of inequalities (4.14) and (4.15).

Appendix B. Detailed Proof of Theorem 2

The optimal house type is specified in equation (4.8), while optimal consumption can be readily ascertained through equation (4.5). Utilizing Itô's formula on the optimal wealth level outlined in equation (4.21), when $y_t^{\lambda_1^*} > \hat{y}$ with $t < \tau^*$, the following applies:

$$\begin{aligned}
\mathrm{d}X_t^* &= \left[-n_1(n_1-1)B\left(y_t^{\lambda_1^*}\right)^{n_1-2} - h_1^{\frac{\gamma-\gamma^*}{\gamma}}\frac{1}{\gamma K}\left(y_t^{\lambda_1^*}\right)^{-\frac{1}{\gamma}-1}\right]\mathrm{d}y_t^{\lambda_1^*} \\
&\quad +\frac{1}{2}\Bigg(-n_1(n_1-1)(n_1-2)B\left(y_t^{\lambda_1^*}\right)^{n_1-3} + h_1^{\frac{\gamma-\gamma^*}{\gamma}}\frac{1+\gamma}{\gamma^2 K}\left(y_t^{\lambda_1^*}\right)^{-\frac{1}{\gamma}-2}\Bigg)\mathrm{d}\left\langle y^{\lambda_1^*}\right\rangle_t \\
&= \Bigg[\left(-n_1(n_1-1)B\left(y_t^{\lambda_1^*}\right)^{n_1-1} - h_1^{\frac{\gamma-\gamma^*}{\gamma}}\frac{1}{\gamma K}\left(y_t^{\lambda_1^*}\right)^{-\frac{1}{\gamma}}\right)(\beta-r) \\
&\quad +\frac{1}{2}\theta^2\left(-n_1(n_1-1)(n_1-2)B\left(y_t^{\lambda_1^*}\right)^{n_1-1} + h_1^{\frac{\gamma-\gamma^*}{\gamma}}\frac{1+\gamma}{\gamma^2 K}\left(y_t^{\lambda_1^*}\right)^{-\frac{1}{\gamma}}\right)\Bigg]\mathrm{d}t \\
&\quad -\theta\left(-n_1(n_1-1)B\left(y_t^{\lambda_1^*}\right)^{n_1-1} - h_1^{\frac{\gamma-\gamma^*}{\gamma}}\frac{1}{\gamma K}\left(y_t^{\lambda_1^*}\right)^{-\frac{1}{\gamma}}\right)\mathrm{d}B_t. \qquad \text{(B.1)}
\end{aligned}$$

Subsequently, comparing the coefficient of the diffusion term $\mathrm{d}B_t$ in (B.1) with those in (3.1) results in the optimal allocation in risky asset:

$$\pi_t^* = \frac{\theta}{\sigma}F_1\left(y_t^{\lambda_1^*}\right).$$

Likewise, for $\bar{y} < y_t^{\lambda_2^*} \leqslant \hat{y}$ with $t < \tau^*$, the optimal investment strategies can be derived in a same manner. Specifically, we have

$$\begin{aligned}
\mathrm{d}X_t^* &= \left[-n_2(n_2-1)A_1\left(y_t^{\lambda_2^*}\right)^{n_2-2} - n_1(n_1-1)A_2\left(y_t^{\lambda_2^*}\right)^{n_1-2} - h_2^{\frac{\gamma-\gamma^*}{\gamma}}\frac{1}{\gamma K}\left(y_t^{\lambda_2^*}\right)^{-\frac{1}{\gamma}-1}\right]\mathrm{d}y_t^{\lambda_2^*} \\
&\quad +\frac{1}{2}\Bigg(-n_2(n_2-1)(n_2-2)A_1\left(y_t^{\lambda_2^*}\right)^{n_2-3} - n_1(n_1-1)(n_1-2)A_2\left(y_t^{\lambda_2^*}\right)^{n_1-3} + h_2^{\frac{\gamma-\gamma^*}{\gamma}}\frac{1+\gamma}{\gamma^2 K}\left(y_t^{\lambda_2^*}\right)^{-\frac{1}{\gamma}-2}\Bigg)\mathrm{d}\left\langle y^{\lambda_2^*}\right\rangle_t \\
&= \Bigg[\left(-n_2(n_2-1)A_1\left(y_t^{\lambda_2^*}\right)^{n_2-1} - n_1(n_1-1)A_2\left(y_t^{\lambda_2^*}\right)^{n_1-1} - h_2^{\frac{\gamma-\gamma^*}{\gamma}}\frac{1}{\gamma K}\left(y_t^{\lambda_2^*}\right)^{-\frac{1}{\gamma}}\right)(\beta-r) \\
&\quad +\frac{1}{2}\theta^2\Bigg(-n_2(n_2-1)(n_2-2)A_1\left(y_t^{\lambda_2^*}\right)^{n_2-1} - n_1(n_1-1)(n_1-2)A_2\left(y_t^{\lambda_2^*}\right)^{n_1-1} + h_2^{\frac{\gamma-\gamma^*}{\gamma}}\frac{1+\gamma}{\gamma^2 K}\left(y_t^{\lambda_2^*}\right)^{-\frac{1}{\gamma}}\Bigg)\Bigg]\mathrm{d}t \\
&\quad -\theta\Bigg(-n_2(n_2-1)A_1\left(y_t^{\lambda_2^*}\right)^{n_2-1} - n_1(n_1-1)A_2\left(y_t^{\lambda_2^*}\right)^{n_1-1} - h_2^{\frac{\gamma-\gamma^*}{\gamma}}\frac{1}{\gamma K}\left(y_t^{\lambda_2^*}\right)^{-\frac{1}{\gamma}}\Bigg)\mathrm{d}B_t. \qquad \text{(B.2)}
\end{aligned}$$

Next, by comparing the coefficient of $\mathrm{d}B_t$ in (B.2) with those in (3.1), the optimal investment in a risky asset is obtained for

$$\pi_t^* = \frac{\theta}{\sigma} F_2\left(y_t^{\lambda_2^*}\right).$$

Once the time t surpasses τ^*, the following condition applies:

$$\begin{aligned} \mathrm{d}X_t^* &= -\kappa^{\frac{1-\gamma}{\alpha\gamma}} h_3^{\frac{\gamma-\gamma^*}{\gamma}} \frac{1}{\gamma K}\left(y_t^{\lambda_3^*}\right)^{-\frac{1}{\gamma}-1} \mathrm{d}y_t^{\lambda_3^*} + \frac{1}{2}\kappa^{\frac{1-\gamma}{\alpha\gamma}} h_3^{\frac{\gamma-\gamma^*}{\gamma}} \frac{1+\gamma}{\gamma^2 K}\left(y_t^{\lambda_3^*}\right)^{-\frac{1}{\gamma}-2} \mathrm{d}\left\langle y^{\lambda_3^*}\right\rangle_t \\ &= \left[-\kappa^{\frac{1-\gamma}{\alpha\gamma}} h_3^{\frac{\gamma-\gamma^*}{\gamma}} \frac{1}{\gamma K}\left(y_t^{\lambda_3^*}\right)^{-\frac{1}{\gamma}} (\beta - r) + \frac{1}{2}\theta^2 \kappa^{\frac{1-\gamma}{\alpha\gamma}} h_3^{\frac{\gamma-\gamma^*}{\gamma}} \frac{1+\gamma}{\gamma^2 K}\left(y_t^{\lambda_3^*}\right)^{-\frac{1}{\gamma}}\right]\mathrm{d}t + \kappa^{\frac{1-\gamma}{\alpha\gamma}} h_3^{\frac{\gamma-\gamma^*}{\gamma}} \frac{1}{\gamma K}\left(y_t^{\lambda_3^*}\right)^{-\frac{1}{\gamma}} \theta \mathrm{d}B_t. \end{aligned}$$

Similarly, employing the analogous methods, we derive the subsequent optimal investment after purchasing a house

$$\pi_{2,t}^* = \frac{\theta}{\sigma} F_3\left(y_t^{\lambda_3^*}\right).$$

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