Incomplete equilibrium model under logarithmic utilities with different time preferences and subjective beliefs-Discrete cash flow model-

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Background and Motivation

- As background, in the QQE (qualitative and quantitative easing) of the Bank of Japan, they purchased long-maturity JGBs (Japanese government bonds) to keep the yield curve of the long end low. (YCC: yield curve control)
- The YCC ended in March 2024, and they announced to decrease purchase amount of long-term bonds. Institutional investors wonder how the yield curve shifts due to the market participants' expectations of the BOJ's purchase plan for their asset liability management.
- Thus, we want to investigate the impact of BOJ's outright purchase plan of JGBs on the yield curve, in other words, how the supply of JGBs affects the yield curve.
- We aim to construct a model to quantify the impact, which incorporates the mechanism such that more/less supply of JGBs against the market demand, leads to lower/higher JGB price and higher/lower yield.

Background and Motivation

- In June 2024, the BOJ announced that it would reduce the purchase amount of JGBs in the future, and the market is wondering how much the BOJ would reduce and what the impact on the yield curve would be.
- The reduction in BOJ's purchase amount indicates a greater supply of JGBs to the market consisting of banks and institutional companies such as asset management companies, insurance companies, and pension funds.
- The biggest interest of the market participants in the Japanese financial market is how the change in the supply and demand of JGBs in the secondary market, i.e., the change in the market outstanding notional of the JGBs, affect the term structure of interest rates would be.

Background and Motivation

- Historically, in 2013, when the outstanding notional of JGBs was 1 quadrillion (1,000 trillion) yen, the BOJ's holding JGB notional was only 100 trillion yen.
- In QQE, led by the former governor Mr. Kuroda, the BOJ aggressively purchased JGBs, and their holding amount is now 600 trillion yen, while the outstanding notional slightly increased to 1.1 quadrillion (1,100 trillion) yen due to chronic financial deficit.
- Thus, by decreasing JGB supply to the market from 900 trillion yen (2013) to 500 trillion yen (2024), the JGB prices went higher, and the JGB yield was once lowered to the minus zone in 2019. (YCC: Yield Curve Control)

(trillion yen)

	BOJ's holding	Secondary Market's holding (Market Out Standing)	Total
2013	100	900	1,000
2024	600	500	1,100

Figure: JGB outstanding notional shares of the BOJ and the market

Background and Motivation

- Now, the BOJ is planning to reduce the annual purchasing amount of 70 trillion yen, which is considered to be equal to the yearly redemption amount of the JGBs the BOJ holds (so the BOJ's JGB position change is +70-70=0).
- The newly issued notional amount of JGBs is 40 trillion yen, except for the reissue against the redemption amount.
- If the BOJ should stop purchasing JGBs (so the BOJ's JGB position change will be 0-70=-70), since the market has to underwrite the reissued JBGs (70 trillion yen) in addition to the newly issued JGBs (40 trillion yen), the outstanding JGB notional in the market increases, that is, more JGB supply in the market, leading to low JGB prices and higher JGB yield.

(trillion yen)

	BOJ's holding	Secondary Market's holding (Market Out Standing)	Total
2025 (if 70 trillion is purchased)	0	40	40
2025 (0 trillion)	-70	110	40

Figure: Change in the JGB outstanding notional of the BOJ and the market when the BOJ's outright purchase amount changes

Background and Motivation

- Below indicates the transition of the market outstanding notional of JGBs, i.e, the outstanding notional held by the secondary market participants excluding the BOJ.
- We observe while the short maturity bonds have been aggressively bought by the BOJ, the market holding of the long term bonds have been increasing, which leads to increased supply, lower bond price and higher interest rates.

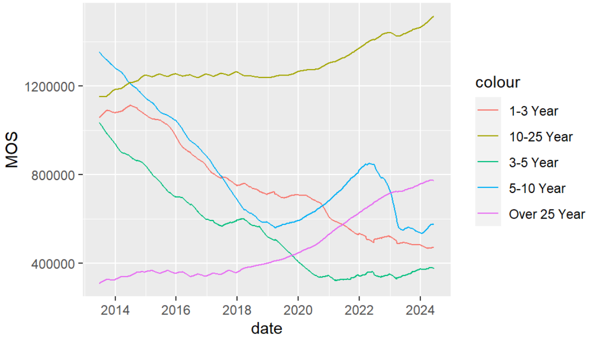

Figure: Market outstanding notional by time-to-maturity

Background and Motivation

- We construct a model to incorporate the mechanism such that more JGB supply in the market leads to lower JGB prices and higher yield.
- In the multi-agent equilibrium model, the institutional investors' consumption in the long end in equilibrium increases as they earn more JGB coupons from increased JGB supply.
- Then, the increase in consumption in the long end implies higher interest in the long end, and we quantify the impact of the BOJ's purchase amount on the yield curve.

Previous literature

- Nakano et al.(2022) estimated the effect of outright purchase of JGBs by the BOJ on the lowering yield curve by an empirical approach by a state space model.
- From a trading perspective, we want to construct a model to endogenously explain how the change in the monetary policy on the BOJ's JGB outright purchase amount affects the yield curve and quantify the impact.

Modeling ideas (discrete cash flow model)

- The problem is that modeling all the details, such as the prices of all maturities of bonds that may be regularly issued, purchased, and redeemed, may be too complicated to deal with the original problem.
- Thus, focusing on the total market value of outstanding bonds in the groups categorized by time to maturity, we construct a discrete-time dividend-paying securities model.

Discrete cash flow model

The model is as follows.

- The model is an incomplete market where the number of Brownian Motions exceeds that of securities (the market value of government bonds categorized by time to maturity).
- The d-dimensional Brownian Motion W may include some economic factors. We assume SDEs of S^j, $j = 1, \ldots, N+1$ where (δ^j is a dividend process) and B is a money market account.

Discrete cash flow model

- First, we suppose a financial market consisting of N zero coupon bonds with different constant maturities, one stock index, and a money market account. Thus, $N+1$ securities and money market account are traded.
- For a trading period $[0, T]$, $T > 0$, let $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{0 \le t \le T}, P)$ be a filtered probability space satisfying the usual conditions. Let W be a d-dimensional Brownian motion where $d > N+1$ and $\{\mathcal{F}\}_{0 \le t \le T}$ be a filtration generated by W.
- Let $\delta^j, r, \mu_S^j, \sigma_S^j$ be $\mathcal{R}$-valued ($\mathcal{R}^{1\times d}$-valued for σ_S^j) $\{\mathcal{F}_t\}$-progressively measurable processes.
- We also suppose that security j, $j = 1, \ldots, N+1$, coupon bonds with time to maturity $\mathcal{T}_j$, $0 < \mathcal{T}_1 < \cdots < \mathcal{T}_N \le T$ and stock index for security $N+1$ pay out δ^j as net redemption or dividends at discrete times $\{t_1, t_2, \ldots, t_K\}$, $0 < t_1 < t_2, \ldots, < t_K \le T$.

Discrete cash flow model

- That is, let D^j be a cumulative dividend process defined as $D^j_t = \sum_{\{k:t_k \le t\}}^{K} \delta^j_{t_k}$, and $dD^j_t := D^j_t - D^j_{t-}$ so that

$$dD^j_t = \delta^j_{t_k}, (t = t_1, \ldots, t_K),\ 0\ (otherwise). \tag{1}$$

- Let $S^j,\ j = 1, \ldots, N$ be the aggregate maker value process of the j-th category of government bonds satisfying an SDE

$$\begin{aligned} dS^j_t &= \mu^j_{S,t} S^j_t dt + S^j_t \sigma^j_{S,t} dW_t - dD^j_t \\ &= r_t S^j_t dt + S^j_t \sigma^j_{S,t} (\theta_t dt + dW_t) - dD^j_t, \end{aligned} \tag{2}$$

where θ is a $\mathcal{R}^d$-valued $\{\mathcal{F}_t\}$-progressively measurable process defined as

$$\theta = \sigma_S^\top (\sigma_S \sigma_S^\top)^{-1} (\mu_{S,t} - r) \in Range(\sigma_S^\top). \tag{3}$$

- Here, $\sigma_S = (\sigma_S^{1,\top}\ \ldots\ \sigma_S^{N+1,\top})^\top$, $\mu_S = (\mu_1 \ldots \mu_{N+1})^\top$, and $Range(\sigma_S^\top)$ is a linear space spanned by $\sigma_S^1\ \ldots\ \sigma_S^{N+1}$.
- Also, we denote the price process of a money market account with instantaneous interest rate r by B, i.e.,

$$B_t = e^{\int_0^t r_s ds}. \tag{4}$$

Modeling ideas (discrete cash flow model)

In this model, we express the total market value of the government bonds in the same zone of time to maturity (say 5-10 years) as the security paying dividends that represent

- the coupon payments,
- redemption of JGBs or outright purchasing JGBs by the BOJ (decreasing of the outstanding market value),
- and reissuance of JGBs by the government (increasing of the outstanding market value).

	Bond (Market Outstanding)	Cash (Market)
New Issue	+	-
Redemption	-	+
Outright Purchase	-	+
Coupon		+

Figure: Impact on the market outstanding JGB notional and the market holding cash (δ in the model)

Modeling ideas (discrete cash flow model)

We remark that the tradable securities are baskets of JGBs in the same category of time to maturity (e.g., 5-10 years) and the agents trade the baskets consisting of the bonds with the same weight existing in the market for their individual optimization.

Modeling ideas (individual optimization problem)

Then, we formulate the following individual optimization problems for agents representing institutional investors.

- We consider a situation where institutional investors, such as insurance companies, asset management, pension funds, invest in government bonds and stocks.
- These institutional investors adjust their positions by monitoring outstanding notional of government bonds categorized by time to maturity and their yields.
- They receive coupons from government bonds, cash from bond redemption, and outright purchasing by BOJ (minus reissuance by the government) and dynamically make optimal consumption and investments.

Individual Optimization Problem

- We suppose that there are I ($I \geq 2$) agents with the log utility on discrete consumption with different beliefs on the Brownian motion.
- Let π^i and $\pi^{i,0}$, $\{\mathcal{F}_t\}$-progressively measurable $\mathcal{R}^{N+1}$-valued ($\mathcal{R}$-valued for $\pi^{i,0}$) process satisfying $\int_0^T |\pi_t^i|^2 ds$, $\int_0^T |\pi_t^{i,0}|^2 ds, < \infty$, $P - a.s.$, be the value basis position of the $N+1$ securities and the money market of agent i, respectively, and X^i be the wealth process of agent i, the total value of agent i's portfolio.
- We also suppose that agent i invests π^i in the $N+1$ securities on a value basis and the rest $\pi^{i,0}$ in the money market account and continuously trades and adjusts its position.

Individual Optimization Problem

- Agent i's utility is log-utility on consumption c^i_k, $k = 1, \ldots, K$ with subjective views η^i. In detail, for the subjective belief of agent i, η^i_T defined as

$$\eta^i_T = \exp\left(\int_0^T \lambda^i_s \cdot dW_t - \frac{1}{2}\int_0^T |\lambda^i_s|^2 ds\right), \tag{5}$$

where λ^i is a $\mathcal{R}^d$-valued $\{\mathcal{F}_t\}$-progressively measurable process satisfying a weak version of Novikov's condition, represents the subjective views on the Brownian motion. Namely, for the probability measure P^i defined as

$$\frac{dP^i}{dP} = \eta^i_T, \tag{6}$$

by Girsanov's theorem, W^{P^i} defined as $dW_t = dW^{P^i}_t + \lambda^i_t dt$ is a P^i-Brownian motion and λ^i indicates the bias of agent i on the Brownian motion W in the physical probability measure.

- Then, each agent solves the optimal consumption and investment problem in an incomplete market setting where the agent optimally consumes and invests to maximize the expected utility of the consumption.

Individual Optimization Problem

- At the discrete times $\{t_1, \ldots, t_K\}$, agent i consumes c_k where $c_k, \ k = 1, \ldots, K$ are $\mathcal{F}_{t_k}$-adapted positive random variables satisfying $E[\sum_{k=1}^{K} c_k^2] < \infty$, that is,

$$X_t^i = \pi_t^{i,0} + \pi_t^i \mathbf{1}, \tag{7}$$

- and the wealth process satisfies the following SDE,

$$\begin{aligned} dX_t^i &= \pi_t^{i,0} \frac{dB_t}{B_t} + \sum_{j=1}^{N+1} \pi_t^{i,j} \frac{dS_t^j}{S_t^j} - dC_t^i \\ &= rX_t^i dt + \pi_i \sigma(\theta_t dt + dW) - dC_t^i, \ X_0^i = x_0^i, \end{aligned} \tag{8}$$

where

$$dC_t^i = C_t^i - C_{t-}^i, \ C_t^i = \sum_{k; t_k \leq t} c_k. \tag{9}$$

Individual Optimization Problem

- Then, we consider the following set of portfolio and consumption processes $((\pi^i, \pi^{i,0}), c^i)$ as

$$\mathcal{A}^i = \{((\pi^i, \pi^{i,0}), c^i) | E[\sum_{k=1}^{K} \frac{Z_{t_k}^{\theta^i}}{B_{t_k}} c_k] \leq x_0, \text{for all risk-neutral probability measures} Z^{\theta^i}\}, \tag{10}$$

- where the risk-neutral probability measure Z^{θ^i} is of the form

$$Z_t^{\theta^i} = e^{\left\{-\frac{1}{2}\int_0^t |\theta_s + \nu_s^i|^2 ds - \int_0^t (\theta_s + \nu_s^i)\cdot dW_s\right\}}; \ \sigma_{S,t}\nu_t^i = 0, \ \forall t \in [0, T], \tag{11}$$

where ν^i is a $\mathcal{R}$-valued $\{\mathcal{F}_t\}$-progressively measurable process satisfying the Novikov's condition.

- The above admissibility condition derives from the following condition: agent i does not go bankrupt, i.e., its wealth is always non-negative.

$$X_t^i \geq 0, \forall [0, T]. \tag{12}$$

Since

$$X_t^i = x_0^i + \int_0^t \pi_s^i \frac{dS_s^c}{S_s} - \sum_{k:t_k < t} c_k \geq 0, \tag{13}$$

taking expectation with the state-price density H^i for $t = T$ yields

$$x_0^i \geq E[\sum_{k=1}^{K} c_k^i H_{t_k}^i]. \tag{14}$$

Individual Optimization Problem

(**Individual Optimization Problem**)

- Maximize

$$\sum_{k=1}^{K} E[\eta^i_T \alpha^i_k \log c^i_k], \tag{15}$$

where $\alpha^i_t = e^{-\beta^i t}$, subject to

$$E[\sum_{k=1}^{K} H^i_k c^i_k] \leq x^i_0, \tag{16}$$

- where $H^i_t = \frac{Z^{\theta^i}_t}{B_t}$, for all possible density process Z^{θ^i} for the risk-neutral probability measure of the following form

$$Z^{\theta^i}_t = e^{\left\{-\frac{1}{2}\int_0^t |\theta_s+\nu^i_s|^2 ds - \int_0^t (\theta_s+\nu^i_s)\cdot dW_s\right\}}; \ \sigma_{S,t}\nu^i_t = 0, \ \forall t \in [0, T], \tag{17}$$

(18)

with respect to $((\pi^i, \pi^{i,0}), c^i)$,

Modeling ideas (market equilibrium)

- We consider market equilibrium, the situation where the total cash going out from the dividend process equals the sum of institutional investors' consumption at each discrete time.
- Particularly, we obtain interest rate and the expected returns of the market value of the bonds in the groups categorized by the time to maturity.
- By shifting exogenously given dividend processes, we observe the impact of the prediction of the outright purchase amount by the BOJ on the yield of each group of bonds categorized by time to maturity.

Market Equilibrium

- We consider a market equilibrium where the total consumption over the I agents equals the sum of dividends over the $N + 1$ securities.
- Then, we obtain the interest rate process r and the market price of risk θ of the securities representing the market value of the government bonds categorized by time to maturity in equilibrium.

Market Equilibrium

(Market Clearing Condition)

- At every discrete time $t_k,\ k = 1, \ldots, K$,

$$\sum_{i=1}^{I} c_k^i = \sum_{j=1}^{N+1} \delta_k^j \equiv \delta_k. \tag{19}$$

When the optimal consumption processes of the agents satisfy this clearing condition, we call that the market is in equilibrium.

Result: Effect of purchasing on the yield curve

- First, we suppose that $\delta^j_k = \delta^j_{t_k}$, $j = 1, \ldots, K$ are realization of the common stochastic process δ^j_t satisfying the following SDE

$$d\delta^j_t = \delta^j_t[\mu^{\delta,j} dt + \sigma^{\delta,j} \cdot dW_t]. \tag{20}$$

- Then, the total dividend over $N+1$ securities $\delta_t = \sum_{j=1}^{N+1} \delta^j_t$ satisfies

$$d\delta_t = \delta_t \left[\left(\sum_{j=1}^{N+1} \frac{\delta^j_t}{\delta_t} \mu^{\delta,j}_t \right) dt + \left(\sum_{j=1}^{N+1} \frac{\delta^j_t}{\delta_t} \sigma^{\delta,j}_t \right) \cdot dW_t \right] \tag{21}$$

$$\equiv \delta_t[\mu^\delta_t dt + \sigma^\delta_t \cdot dW_t], \tag{22}$$

$$\tag{23}$$

where

$$\mu^\delta_t := \frac{1}{\delta_t} \left(\sum_{j=1}^{N+1} \delta^j_t \mu^{\delta,j}_t \right); \ \sigma^\delta_t := \frac{1}{\delta_t} \left(\sum_{j=1}^{N+1} \delta^j_t \sigma^{\delta,j}_t \right). \tag{24}$$

Result: Effect of purchasing on the yield curve

Then, we obtain the following interest rate process, the market price of risk, the volatility processes and the expected return processes of the securities in equilibrium.

Proposition

A continuous interest rate process r

$$r_t = +\mu_t^\delta + \sigma_t^\delta \frac{\sum_i \hat{\lambda}_t^i \frac{\alpha_t^i Z_t^i}{y^i}}{\sum_i \frac{\alpha_t^i Z_t^i}{y^i}}, \tag{25}$$

and the market price of risks θ

$$\theta_t = \sigma_t^\delta - \frac{\sum_i \hat{\lambda}_t^i \frac{\alpha_t^i Z_t^i}{y^i}}{\sum_i \frac{\alpha_t^i Z_t^i}{y^i}}, \tag{26}$$

where $y^i = \frac{\sum_{k=1}^K \alpha_k^i}{x_0^i}$, $Z_t^i = \exp\left(-\frac{1}{2}\int_0^t |\hat{\lambda}_s|^2 ds + \int_0^t \hat{\lambda}_s dW_s\right)$, $Z_t^\theta = \exp\left(-\frac{1}{2}\int_0^t |\theta_s|^2 ds - \int_0^t \theta_s dW_s\right)$.
Here, $(\hat{\lambda})^\perp$ *is an orthogonal part of* λ^i *to the linear space spanned by* σ_S*, i.e.,* $\lambda^i = \hat{\lambda}^i + (\hat{\lambda}^i)^\perp$, $\hat{\lambda}^i \in Range(\sigma_S^\top), \sigma_S(\hat{\lambda}^i)^\perp = 0$.

Result: Effect of purchasing on the yield curve

- By the news or market expectation predicting decrease/increase in the purchase amount of JGBs by BOJ, while the front end divided corresponding to the outright purchase decreases/increases, the long end dividends corresponding to the coupon payments and redemption in the future increase/decrease.
- Then, the securities' yield corresponding to the government bonds' market value categorized by time to maturity goes higher/lower.

Result: Individual effect of purchasing specific zone

Proposition

Moreover, μ_S^j the expected return process of S^j in equilibrium is expressed as

$$\mu_{S,t}^j = r_t + \sigma_{S,t}^j \theta_t, \tag{27}$$

where the volatility process σ_S^j in equilibrium are expressed as follows.

$$\sigma_{S,t}^j = \theta_t + \frac{\sum_{k:t_k>t} \delta_t^j \frac{Z_t^\theta}{B_t}}{\sum_{k:t_k>t} \delta_t^j \frac{Z_t^\theta}{B_t} + \sum_{k:t_k>t} \int_t^{t_k} E[(\mu_s^{\delta,j} - r_s - \sigma_s^{\delta,j}\theta_s)|\mathcal{F}_t]ds}(\sigma_t^{\delta,j} - \theta_t)$$
$$+ \frac{\sum_{k:t_k>t} \int_t^{t_k} E[D_t(\mu_s^{\delta,j} - r_s + \sigma_s^{\delta,j}\theta_s)|\mathcal{F}_t]ds}{\sum_{k:t_k>t} \delta_t^j \frac{Z_t^\theta}{B_t} + \sum_{k:t_k>t} \int_t^{t_k} E[(\mu_s^{\delta,j} - r_s - \sigma_s^{\delta,j}\theta_s)|\mathcal{F}_t]ds} \tag{28}$$

Result: Individual effect of purchasing specific zone

- In the expression of μ_S^j, with the market price of risk in equilibrium is given by (26), μ_S^j is rewritten as

$$\begin{aligned} \mu_{S,t}^j &= r_t + \sigma_{S,t}^j \theta_t \\ &= r_t + \sigma_{S,t}^j \sigma_t^\delta - \sigma_{S,t}^j \frac{\sum_{i=1}^I \hat{\lambda}_t^i \frac{\alpha_t^i Z_t^i}{y^i}}{\sum_{i=1}^I \frac{\alpha_t^i Z_t^i}{y^i}}. \end{aligned} \tag{29}$$

- This implies that if S^j and δ are positively correlated, the term $+\sigma_S^j \sigma^\delta$ works as a premium of the expected return μ_S^j over the interest rate r, which holds true by the expression of S^j.

Result: Individual effect of purchasing specific zone

- Specifically, the volatility process σ_S^j in equilibrium is given as follows, when the Malliavin derivative part is zero, i.e.,

$$\sigma_{S,t}^j = + \frac{\sum_{\{k:T_k \geq t\}} \frac{Z_t^\theta}{B_t} \delta_t^j}{\sum_{\{k:T_k \geq t\}} \frac{Z_t^\theta}{B_t} \delta_t^j + \sum_{\{k:T_k \geq t\}} E[\int_t^{t_k} (\mu_s^{\delta,j} - r_s - \sigma_s^{\delta,j} \theta_s) | \mathcal{F}_t] ds} \sigma_t^{\delta,j}$$
$$+ \left(\frac{\sum_{\{k:T_k \geq t\}} E[\int_t^{t_k} (\mu_s^{\delta,j} - r_s - \sigma_s^{\delta,j} \theta_s) | \mathcal{F}_t] ds}{\sum_{\{k:T_k \geq t\}} \frac{Z_t^\theta}{B_t} \delta_t^j + \sum_{\{k:T_k \geq t\}} E[\int_t^{t_k} (\mu_s^{\delta,j} - r_s - \sigma_s^{\delta,j} \theta_s) | \mathcal{F}_t] ds} \right) \theta_t. \quad (30)$$

- This expression of σ_S^j particularly indicates that if the inner product between σ^δ and the leading term $\sigma^{\delta,j}$ increases/decreases, the expected return μ_S^j goes higher/lower due to the term.

Result: Individual effect of purchasing specific zone

- Since

$$
\begin{aligned}
&\sigma_t^{\delta,j}\sigma_t^{\delta} \\
&= \sigma_t^{\delta,j}\left(\sum_{l=1}^{N+1}\frac{\delta_t^l}{\delta_t}\sigma_t^{\delta,l}\right) \\
&= \frac{\delta_t^j}{\delta_t}|\sigma_t^{\delta,j}|^2 + \sum_{l\neq j}\frac{\delta_t^l}{\delta_t}\sigma^{\delta,j}\sigma_t^{\delta,l},
\end{aligned}
\tag{31}
$$

- this is particularly true at the long end when $\frac{\delta_t^j}{\delta_t}$, the weight of δ^j in δ becomes high/low such as when the central bank buys back small/large amount of bonds in the front end and more/less amount of bonds are redeemed in the long end.

Conclusion

- We have modeled institutional investors' optimal portfolio and consumption problem by focusing on the market outstanding value of the government bonds in the same zone categorized by time to maturity.
- We have obtained equilibrium interest and yields of the government bonds in the zones categorized by time-to-maturity.
- The model incorporates the change in the market supply of JGBs and quantifies the effect on the yield curve.
- The model is useful for observing the impact of the BOJ's purchase of JGBs on the yield curve, which the market is closely monitoring.

References

- Kizaki, K., Saito, T., & Takahashi, A. (2024). Equilibrium multi-agent model with heterogeneous views on fundamental risks. Automatica, 160, 111415.
- Kizaki, K., Saito, T., & Takahashi, A. (2024). A multi-agent incomplete equilibrium model and its applications to reinsurance pricing and life-cycle investment. Insurance: Mathematics and Economics, 114, 132-155.
- Nakano, M., Takahashi, A., Takahashi, S., & Tokioka, T. (2018). On the effect of Bank of Japan's outright purchase on the JGB yield curve. Asia-Pacific Financial Markets, 25, 47-70.