Are Factor Pricing Models Internally Consistent?

Hankil Kang (Dankook Univ.)
Doojin Ryu (Sungkyunkwan Univ.)
Robert I. Webb (Univ. of Virginia)

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Motivation: Anomalies and Asset Pricing Models

- Modern finance theory owes much to empirical analysis.
- The notion of market efficiency came from observations that changes in speculative prices appeared to follow a "random walk." Samuelson's theory came later.
- The discovery by Rolf Banz [1981] that the risk-adjusted returns on small cap stocks exceed those for large cap stocks—the size effect- challenged the validity of the Capital Asset Pricing Model and/or market efficiency.
- The subsequent discovery of other seeming "anomalies" led to the discovery of many potential risk factors as well as new models of asset pricing.

Time Series Analysis and Structural Econometric Models

- There are two principal ways of describing the behavior of changes in speculative prices:
  - Time series analysis
  - Structural Econometric Model(s)
- In most cases it is easier to estimate the time series than build a structural econometric model.
- It is important to recognize that these two approaches are equivalent. They should give the same solution.
- Changes in speculative prices follow very simple time series processes. This observation, as noted earlier, led to the notion that speculative markets are efficient.

Time Series Analysis and Structural Econometric Models

- Asset pricing models are structural econometric models.
- Available information should be used. This includes the information about the time series behavior of the output variables from proposed SEMs.
- Thus, information in the time series behavior of changes in speculative prices can be used in building SEMs. This is the essence of the Zellner-Palm consistency contraints
- Time series behavior of changes in speculative prices is NOT controversial. The appropriate factor pricing model is very controversial.

Motivation

- Asset pricing in the stock market
  - From the 30 years from Fama and French (1993), it can be summarized by anomalies and asset pricing factors.
- Anomalies vs. factors
  - Anomalies: patterns of stock returns that cannot be explained by existing asset pricing models
  - Factors: systematic components that explain as many existing anomalies as possible
- Hou, Xue, and Zhang (2020)
  - Covering 452 anomalies
  - “Only” 158 anomalies are significant

Motivation

- Factor zoo phenomenon
  - CAPM
  - Fama-French three factors: (MKT, SMB, HML)
  - Carhart four factors: FF3+WML
  - Fama-French five factors: FF3 + RMW, CMA
  - Hou, Xue, Zhang (2015) four, five factors: MKT, ME, INV, ROE + EG
  - …and many other factors

Motivation

- Most factor models are linear
  - Linear stochastic discount factors
  - Multi-beta models
  - Focusing on alphas from time-series regressions

- Ross (1976)'s APT
  - $E(r_i) - r_f = \lambda_0 + \beta_{i1}\lambda_1 + \beta_{i2}\lambda_2 + \cdots + \beta_{iK}\lambda_K$
  - $\lambda_j = E(r_j) - r_f$, r_j is the mimicking portfolio return for factor j

- Time-series form of the APT
  - $r_{it} - r_{ft} = \alpha_i + \beta_{i2}(r_{1t} - r_{ft}) + \cdots + \beta_{iK}(r_{Kt} - r_{ft}) + \epsilon_{it}$
  - We focus on this time-series data generating process.

Summary

- We revisit Zellner-Palm (1974) consistency
  - Zellner and Palm (1974): In a broad structural models with ARIMA time series, the RHS and the LHS should have consistent ARIMA structure.

- Webb (1985, 1990): the implication of Zellner-Palm consistency on asset pricing theory
  - Asset returns follow simple (random walk or IMA (1)) time-series process.
  - If the output variable (asset returns) follow a simple time-series process, the input variables (asset pricing factors) should also follow a simple ARIMA process.
  - The more complex the model is, the less probable it passes the Zellner-Palm consistency test and the less likely it is correct.

Preliminary Results

- We test whether various proposed risk factors for stock (asset) market returns follow simple ARMA structures. That is, are the various proposed asset pricing models consistent with the observed time series behavior of the output variables (i.e., changes in speculative prices)?
- Univariate tests on famous risk factors
  - The Fama-French five factors follow very simple ARMA processes.
  - Some of the HXZ five factors (size, expected growth) follow relatively complex ARMA processes (ARMA (2,2)).
- Univariate tests on 153 factors by Jensen, Kelly, and Pedersen (2023)
  - We report $p + q$ from an ARMA (p,q) models for factors.
  - 52 out of 153 factors have $p + q \geq 4$.

Potential Contributions

- The use of the Zellner-Palm consistency constraints
  - Suggests that the correct model of asset pricing is unlikely to be "too complex" or have many factors in potential linear models.
  - Allows one to screen potential risk factors using relatively simple empirical tests.
  - Provides results consistent with Brzygalova et al. [2023] who employed over two quadrillion tests on a smaller set of potential risk factors.

Potential Limitations

- The limitations of our analyses include:
  - It does not work on non-linear models (e.g. consumption-based models). However, most asset pricing models are linear.
  - We do not answer the question: which factors are better (right, more effective, pricing well).

Related Literature

- Maio and Santa-Clara (2012)
  - Tests for consistency with the ICAPM
  - The ability of factors to forecast future investment opportunity set
- Harvey, Liu, and Zhu (2016)
  - Suggest t-statistics over 3.
- Bryzgalova, Huang, and Julliard (2023)
  - Bayesian tests for linear factor models – the stochastic discount factor approach
  - Run over two quadrillion potential models

Zellner-Palm Consistency Constraints

- A two-factor APT model:
  - $r_t = v_1(B)X_{1t} + v_2(B)X_{2t} + n_t$
  - r_t: the output return with ARMA (p_0, q_0)
  - X_{it}: two factors with seasonal ARMA processes $[(p_i, q_i) \times (p^s, q^s)S]$
  - $v_i(B) = \frac{n_i(B)}{d_i(B)}$: transfer functions (orders of $n_i(B)$ and $d_i(B)$: s_i, r_i)
  - n_t: noise with ARMA (p_n, q_n) process
  - b: delay operator (lag in response r_t to changes in the X_t)

Zellner-Palm Consistency Constraints

- The Zellner-Palm consistency constraints:
  - AR term: $p_0 \; vs. \; p_n + r_1 + p_1 + p_1^s + p_2 + p_2^s$
  - MA term: $q_0 \; vs. \max \begin{bmatrix} (p_n + b + r_2 + p_2 + p_2^s + q_1 + q_1^s + s_1), \\ (p_n + b + r_1 + p_1 + p_1^s + q_2 + q_2^s + s_2), \\ (b + r_2 + p_2 + p_2^s + r_1 + p_1 + p_1^s + q_n) \end{bmatrix}$
  - Let's make it simpler!

Zellner-Palm Consistency Constraints

- No seasonality: $p_i^s, q_i^s = 0$
- No lag: $b = 0$
  - AR term: $p_0\ vs. p_n + r_1 + p_1 + p_2$
  - MA term: $q_0\ vs. \max\begin{bmatrix}(p_n + r_2 + p_2 + q_1 + s_1),\\ (p_n + r_1 + q_2 + s_2),\\ (r_2 + p_2 + r_1 + p_1 + q_n)\end{bmatrix}$
- No transfer function: $s_i, r_i = 0$
  - AR term: $p_0\ vs.\ p_n + p_1 + p_2$
  - MA term: $q_0\ vs. \max\begin{bmatrix}(p_n + p_2 + q_1),\\ (p_n + p_1 + q_2),\\ (p_2 + p_1 + q_n)\end{bmatrix}$

Zellner-Palm Consistency Constraints

- No ARMA structure in the noise: $p_n = q_n = 0$
  - AR term: $p_0 \; vs. \; p_1 + p_2$
  - MA term: $q_0 \; vs. \max \begin{bmatrix} (p_2 + q_1), \\ (p_1 + q_2), \\ (p_1 + p_2) \end{bmatrix}$

- Univariate case:
  - AR term: $p_0 \; vs. \; p_1$
  - MA term: $q_0 \; vs. \max[p_1, q_1]$

- Our empirical question for univariate analysis:
  - Are the orders of the ARMA(p,q) model “too high”?
  - Currently, we set the hurdle by $p + q$.

Empirical Tests

- We test whether any of the numerous potential stock market risk factors follow simple times series process (in terms of ARIMA process).

- ARIMA estimation for famous risk factors
  - Sample period: 1967.01~2023.12
  - FF5, HXZ q5, momentum

- ARIMA estimation for Jensen, Kelly, and Pedersen (2023) 153 factors
  - Sample period: 1971.11~2023.12

Univariate Results

- ARIMA estimation for FF5, HXZ q5, and momentum
  - FF5 factors (including MKT) follow quite simple processes.
  - Size (HXZ), ROE (HXZ), and EG (HXZ) are doubtable.

	p	d	q
MKT	1	0	0
SMB	1	0	0
HML	1	0	0
RMW	0	0	1
CMA	1	0	0
R_ME	2	0	2
R_IA	1	0	1
R_ROE	0	0	3
R_EG	2	0	2
MOM11_1	1	0	0

Univariate Results

- Results for ARIMA (p,d,q) model
  - We test 153 Jensen, Kelly, Pedersen (2023) factors.
  - 81 factors satisfy $p + q \leq 1$. (53%)
  - 52 factors: $p + q \geq 4$ (34%)
  - 64 factors: $p + q \geq 3$. (42%)

p+q	Number of factors	Percentage
0	39	25.5
1	42	27.5
2	8	5.2
3	12	7.8
4	32	20.9
5	12	7.8
6	5	3.3
7	3	2.0
Total	153	100

Conclusions

- Time series analysis and structural econometric model building are equivalent approaches to describing the behavior of changes in speculative prices.
- Most financial economists agree that changes in specula -tive prices follow simple time series processes.
- There is widespread controversy over the appropriate model of asset pricing. So many risk factors have been proposed that there is the problem of a “factor zoo”.
- Application of the Zellner-Palm consistency constraints is a very simple approach to limiting potential valid *linear* models of asset pricing.

Thank You

Any comments are welcome.