Liability-Driven Investment under Inflation Risk*

Bong-Gyu Jang
Department of Industrial and Management Engineering, POSTECH, 37673, 77 Cheongam-Ro, Nam-Gu, Pohang City, Geongbuk, Republic of Korea
E-mail: bonggyujang@postech.ac.kr

Hyeontae Jo
Department of Industrial and Management Engineering, POSTECH, 37673, 77 Cheongam-Ro, Nam-Gu, Pohang City, Geongbuk, Republic of Korea
E-mail: jht6448@postech.ac.kr

Myung Jun Kim
Department of Industrial and Management Engineering, POSTECH, 37673, 77 Cheongam-Ro, Nam-Gu, Pohang City, Geongbuk, Republic of Korea
E-mail: kmj4720@postech.ac.kr

*Preliminary versions of the paper were presented at the 2024 KIIE-KORMS Conference and we would like to thank Byung Hwa Lim and Yong Hyun Shin for their valuable and constructive comments. This work was supported by the Ministry of Education of the Republic of Korea and the National Research Foundation of Korea.(NRF-2023R1A2C2003927)

Liability-Driven Investment under Inflation Risk

Abstract

This paper introduces a novel model for liability-driven investment, specifically addressing inflation risk. Framed as a portfolio selection model with a Value-at-Risk (VaR) constraint, our model aids fund managers in maximizing their utility of the funding ratio between assets and liabilities. Exploiting an option-based approach, we derive the optimal investment strategy in closed form. Using carefully selected parameters, we show that higher expected inflation rates prompt a greater allocation to both classes of risky assets: stocks and inflation-linked index bonds. This effect becomes more pronounced as the correlation between the two risky asset classes decreases. Additionally, we find that the fund manager's optimal investment strategies should change significantly based on the current funding ratio, with a shorter investment period resulting in greater variability in these strategies.

Key Words: Asset Allocation, Liability-driven Investment, Pension Fund, Inflation Risk
JEL Classification: E31, G11, G23

1 Introduction

In the dynamic and rapidly evolving world of finance, it has become increasingly important to manage a portfolio that meets a fund manager's liabilities while mitigating various economic risks. The COVID-19 pandemic, in particular, has caused significant changes in the economic context, leading to a surge of interest in liability-driven investment (LDI) strategies.

LDI strategies have diverged from conventional portfolio management paradigms, prioritizing the management of liability risks. Traditionally, these strategies have been adopted by pension funds and insurance companies with significant, specified future obligations. The conceptual framework of LDI is grounded in the objective of meeting financial liabilities by precisely managing assets and aligning them with the temporal and financial horizons of the liabilities.

The COVID-19 pandemic has further intensified the challenges and risks faced by the investment sector, particularly through its impact on inflation. The steady rise in the overall price level of goods and services erodes purchasing power and distorts expected investment returns. Consequently, it is crucial to adopt strategic measures to navigate these uncertain economic conditions and safeguard the feasibility of LDI strategies.

This study aims to unravel the complexity of LDI strategies in light of inflation risks. We present a comprehensive analysis of practical approaches that combine liability management with strategic inflation hedging. This study offers a detailed examination of LDI in the context of inflation risk, incorporating analytical models and theoretical foundations. It demonstrates a reliable investment strategy of pension fund managers to mitigate the impact of inflation risk on their portfolio.

Previous research, such as Grossman and Zhou (1996), has addressed the optimal portfolio selection problem under the constraint of terminal wealth, showing that this restriction can affect both the optimal wealth level and asset allocation. Additionally, Basak and Shapiro (2001) examined dynamic portfolio selection problems in the presence of both Value-at-Risk (VaR) and Expected Shortfall (ES) constraints. Their findings indicate that VaR might lead to a larger exposure to risky assets, while ES can address the limitations associated with relying solely on VaR. Based on their work, we employ the VaR constraint on the terminal wealth of pension fund managers in our model.

As previously mentioned, the importance of LDI has spurred a significant amount of research. Sharpe and Tint (1990) emphasized the need to simultaneously consider liabilities and wealth, utilizing a mean-variance portfolio model with liability asset ratios. Building on Sharpe's work, Detemple and Rindisbacher (2008) introduced a dynamic model with respect to the funding ratio, the ratio between the fund's assets and liabilities, and find the optimal investment strategies in the presence of a shortfall constraint. Ang et al. (2012) addressed the mean-variance portfolio optimization problem with utility function constraints, treating it as a European put option, but did not consider the dynamically-readjusted portfolio.

Regarding inflation, Fischer (1975) analyzed the demand for inflation-linked index bonds by solving households' optimization problems, taking inflation risk into account. He highlighted the significance of such indexed bonds and explained the rationale for introducing an inflation-linked index bond market. Brennan and Xia (2002) tackled a dynamic asset allocation issue that included stochastic interest rates and indexed bonds, demonstrating the importance of inflation-linked index bonds for conservative and long-term investments.

When implementing LDI strategies, fund managers typically adopt a long-term investment horizon. Given the significant impact of inflation risk in this context, it is crucial to consider its effects when employing LDI strategies. However, there is currently no research that examines the combined effects of LDI and inflation risk.

In this paper, we propose a portfolio selection model for LDI that considers inflation risk. The assets included in this model are typical risky assets such as stocks, (almost) risk-free assets such as money market funds, and inflation-linked index bonds, such as US Treasury Inflation-Protected Securities (TIPS). The stochastic process governs the evolution of all assets, except for money market funds. Similarly, liabilities are assumed to follow a time-varying, but deterministic, process. The model incorporates a VaR constraint on the funding ratio, which aligns well with the concept of LDI.

Using carefully selected parameters, we show that higher expected inflation rates prompt a greater allocation to both classes of risky assets: stocks and inflation-linked index bonds. This effect becomes more pronounced as the correlation between the two risky asset classes decreases. Additionally, we find that the fund manager's optimal investment strategies should change significantly based on the current funding ratio, with a shorter investment period resulting in greater variability in these strategies.

Furthermore, we extend the option-based approach outlined in Kraft and Steffensen (2013) and Jang and Park (2016) to accommodate multiple risky assets. This enhancement facilitates the derivation of a closed-form optimal investment strategy, a task acknowledged for its inherent difficulty in portfolio selection problems. This constitutes our technical contribution to the existing literature.

The paper is structured as follows. Section 2 outlines the market economy considered in this study, while Section 3 presents the optimization problem and its analytic solutions. Section 4 showcases notable empirical results, and Section 5 provides concluding remarks for this paper.

2 Financial Market

In the context of portfolio theory, two asset classes, so-called risk-free and risky assets, have been employed for the financial market. In our paper, however, we employ an additional asset class, which can be a hedging tool against inflation risk, in order to assess how inflation risk affects the investment behaviors of market participants. Brennan and Xia (2002) suggest that the consumer price index (CPI) is used as the (commodity) price level of inflation rates and assumes that it is expressed as a diffusion process.

Following their idea, we assume that the price level of the inflation rate, I_t, is evolved by

$$\frac{dI_t}{I_t} = \mu_I dt + \sigma_I dW_{I,t}$$

where μ_I is the instantaneous expected inflation rate, σ_I is its volatility, and $W_{I,t}$ is a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We employ an inflation-linked index bond (IIB), which its price P_t at time t evolves as follows:

$$\frac{dP_t}{P_t} = Rdt + \frac{dI_t}{I_t} = (R + \mu_I)dt + \sigma_I dW_{I,t},$$

where R is the real interest rate. We assume that the inflation-linked index bond has exactly the same risk source with the inflation risk.

We assume that the risky asset class comprises a total of n $(n \geq 1)$ stocks, $S_{i,t}$ $(i \in \{1, 2, \cdots, n\})$, and the risk-free asset class has a risk-free asset, B_t. Assume their price level processes are

$$\begin{cases} \dfrac{dS_{i,t}}{S_{i,t}} = \mu_i dt + \sigma_i dW_{i,t}, \\ \dfrac{dB_t}{B_t} = rdt, \end{cases}$$

where μ_i $(\mu_i \geq r)$ is the expected rate of return for stock i, and σ_i is their volatility, r is the instantaneous nominal risk-free rate.

We impose the following correlation structure on the three asset classes:

$$\begin{cases} d\langle W_{i,t}, W_{j,t}\rangle = \rho_{i,j} dt \;\text{ for } i, j \in \{1, 2, \cdots, n\} \text{ and } i \neq j, \\ d\langle W_{i,t}, W_{I,t}\rangle = \rho_{i,I} dt \;\text{ for } i \in \{1, 2, \cdots, n\}. \end{cases}$$

Here, $\rho_{i,j}$ and $\rho_{i,I}$ are assumed to be in the interval $(-1, 1)$ for all $i, j \in \{1, 2, \cdots, n\}$.

3 Model

3.1 Fund Manager's Problem

One possible way to consider the liability-driven investment (LDI) in investment decision making is to consider the funding ratio as a target variable. We first clarify the concepts of fund manager's asset holding and liability at each time spot. Suppose that $\pi_{i,t}$ $(i \in \{1, 2, \cdots, n\})$ denote the proportion of assets held in stock i and $\pi_{I,t}$ denote that in an inflation-linked index bond. Then, the fund manager's wealth X_t at time t evolves by following

$$\frac{dX_t}{X_t} = \sum_{i=1}^{n} \pi_{i,t} \frac{dS_{i,t}}{S_{i,t}} + \pi_{I,t} \frac{dP_t}{P_t} + \Big(1 - \sum_{i=1}^{n} \pi_{i,t} - \pi_{I,t}\Big) \frac{dB_t}{B_t}.$$

We assume the fund manager's liability is deterministic over time.[1] Therefore, we exploit an integrable function $f(t)$ to express the liability L_T as deterministic process:

$$dL_t = f(t)dt,\; L_0 = l_0,\; \text{and } l_0 + \int_0^t f(s)ds > 0 \text{ for all } t. \tag{1}$$

[1]The assumption does not fully reflect the stochastic nature of changing liability characteristics, but the determined liability value can be considered as the expected value of future liability. Sharpe and Tint (1990) and Ang et al. (2012) and some others assume the liability follows a stochastic process. However, these studies were limited by their inability to solve the portfolio choice problem analytically in a continuous time. Our model has strengths in this regard.

The funding ratio F_t can now be defined as the ratio between the fund manager's total wealth and liability:

$$F_t = \frac{X_t}{L_t}.$$

and it follows

$$\begin{aligned}\frac{dF_t}{F_t} = &\Big(\sum_{i=1}^{n} \pi_{i,t}(\mu_i - r) + \pi_{I,t}(\mu_I + R - r) + r - \frac{f(t)}{l_0 + \int_0^t f(s)ds}\Big)dt \\ &+ \sum_{i=1}^{n} \sigma_i \pi_{i,t} dW_{i,t} + \sigma_I \pi_{I,t} dW_{I,t}.\end{aligned} \tag{2}$$

If the funding ratio is low, there is a shortage of assets compared to liabilities, indicating a potential risk in asset and liability management (ALM) for the fund manager. In the context of pension funds, it is reasonable to impose some constraints on the funding ratio to mitigate this risk. Ang et al. (2012) also demonstrates the feasibility of controlling liability by limiting the funding ratio, as evidenced by several countries implementing systematic funding ratio restrictions. On the other hand, a high funding ratio implies a huge wealth compared to liabilities, and it is reasonable that investments are made in the direction of maximizing the fund manager's utility. To be precise, the objective of our model is to maximize the expected utility of the funding ratio.

We assume that the fund manager has a utility preference of constant relative risk aversion (CRRA) type, and thus, our problem can be stated as a problem to find the following value function V by controlling her investment strategy $\mathbf{\Pi}_t = (\pi_{1,t}, \pi_{2,t}, ..., \pi_{n,t}, \pi_{I,t})^\intercal$: for the investment horizon $T \in (0, \infty)$, the current time $0 < t < T$, and current wealth x,

$$V(t,x) = \max_{\mathbf{\Pi}_t} E_t \left[\frac{F_T^{1-\gamma}}{1-\gamma} \middle| X_t = x \right],$$

where γ ($\gamma > 0,\ \gamma \neq 1$) is the coefficient of relative risk aversion, $E_t[\cdot]$ is the conditional expectation at time t under $\mathbb{P}-$measure. We impose a value-at-risk (VaR) constraint on the funding ratio: for $k \geq 0$ and $0 < \alpha \leq 1$,

$$\mathbb{P}(F_T > k) \geq 1 - \alpha. \tag{3}$$

3.2 Benchmark Case without the VaR Constraint

We examine the case where there exists no VaR constraint (call it the 'benchmark' case) to compare it with our problem. Without any VaR constraint (3), the Bellman equation is simply

$$\begin{cases} 0 = \max\limits_{\Pi_t} \Big\{ \frac{\partial V}{\partial t} + \Big(r + \sum_{i=1}^{n} \pi_{i,t}(\mu_i - r) + \pi_{I,t}(\mu_I + R - r) \quad - \frac{f(t)}{l_0 + \int_0^t f(s)ds} \Big) F_t \frac{\partial V}{\partial F} \\ \qquad\qquad + \frac{1}{2}\frac{\partial^2 V}{\partial F^2} \mathbf{\Pi}_t^\intercal \mathbf{\Sigma} \mathbf{\Pi}_t F^2 \Big\}, \\ V(T,x) = \frac{(x/L_T)^{1-\gamma}}{1-\gamma}. \end{cases}$$

where $\mathbf{\Sigma} = (\sigma_{i,j})_{i,j \in \{1,2,\cdots,n,n+1\}}$ denotes the covariance matrix of n stocks and the inflation-linked index bond (it is the $(n+1)$-th asset corresponding to the $(n+1)$-th row and column of the covariance matrix). The first-order condition reveals the optimal proportion $\mathbf{\Pi}$ as

$$\mathbf{\Pi}_t^* = -\frac{\mathbf{\Sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1}_{n+1})(\partial V/\partial F)}{F_t(\partial^2 V/\partial F^2)},$$

where $\boldsymbol{\mu} = (\mu_1, \mu_2, ..., \mu_n, \mu_I + R)^\intercal \in \mathbb{R}^{(n+1)\times 1}$, $\mathbf{1}_{n+1} = (1, ..., 1)^\intercal \in \mathbb{R}^{(n+1)\times 1}$. The value function can be expressed through a deterministic function h with $h(T) = 1$, as seen in the works of Merton (1969, 1971) and Kraft and Steffensen (2013):

$$V(t, F_t) = h(t)\frac{F_t^{1-\gamma}}{1-\gamma}.$$

The optimal proportion can be rewritten as

$$\mathbf{\Pi}_t^* = \frac{\mathbf{\Sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1}_{n+1})}{\gamma}. \tag{4}$$

Thus, the funding ratio evolution equation in (2) becomes

$$dF_t = \Big(r + \frac{\boldsymbol{\theta}^\intercal\boldsymbol{\theta}}{\gamma} - \frac{f(t)}{l_0 + \int_0^t f(s)ds}\Big)F_t dt + \frac{\boldsymbol{\phi}^\intercal}{\gamma}F_t d\mathbf{W}_t, \quad F_0 = f_0 > 0 \tag{5}$$

where $\boldsymbol{\theta} = \mathbf{\Sigma}^{-1/2}(\boldsymbol{\mu} - r\mathbf{1}_{n+1})$, $\boldsymbol{\phi} = \mathbf{\Gamma}\mathbf{\Sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1}_{n+1}) = (\phi_1, ..., \phi_{n+1})^\intercal$ and $\mathbf{W}_t = (W_{1,t}, ..., W_{n,t}, W_{I,t})^\intercal$. Here, $\mathbf{\Gamma} \in \mathbb{R}^{(n+1)\times(n+1)}$ denotes the diagonal matrix with its (i, i)-th element $\mathbf{\Gamma}_{i,i} = \sigma_i$ for $i \in \{1, 2, \cdots, n\}$ and $\mathbf{\Gamma}_{n+1,n+1} = \sigma_I$.

3.3 The Case in the Presence of the VaR Constraint: A Solution

Based on the option-based approach introduced by Kraft and Steffensen (2013), we develop its multi-dimensional version. To the end, we first define the $(n + 1)$ mutually-independent risky assets, $A_{i,t}$ $(i \in \{1, 2, \cdots, n + 1\})$, by exploiting the original $(n + 1)$ risky assets[2]: for standard Brownian motions $\widetilde{W}_{i,t}$ $(i \in \{1, 2, \cdots, n + 1\})$,

$$\frac{dA_{i,t}}{A_{i,t}} = \widetilde{\mu}_i dt + \widetilde{\sigma}_i d\widetilde{W}_{i,t},$$

with

$$\begin{cases} d\langle \widetilde{W}_{i,t}, \widetilde{W}_{i,t}\rangle = dt \ \text{ for } i \in \{1, 2, \cdots, n + 1\}, \\ d\langle \widetilde{W}_{i,t}, \widetilde{W}_{j,t}\rangle = 0 \ \text{ for } i, j \in \{1, 2, \cdots, n + 1\}, i \neq j. \end{cases}$$

Then from (5), the funding ratio process Y_t for the unconstrained case can be written as[3]

$$dY_t = \Big(r + \frac{\widetilde{\boldsymbol{\theta}}^\intercal\widetilde{\boldsymbol{\theta}}}{\gamma} - \frac{f(t)}{l_0 + \int_0^t f(s)ds}\Big)Y_t dt + \frac{\widetilde{\boldsymbol{\theta}}^\intercal}{\gamma}Y_t d\widetilde{\mathbf{W}}_t, \quad Y_0 = y_0 > 0 \tag{6}$$

where $\widetilde{\boldsymbol{\theta}} = \widetilde{\mathbf{\Sigma}}^{-1/2}(\widetilde{\boldsymbol{\mu}} - r\mathbf{1}_{n+1}) = (\widetilde{\theta}_1, ..., \widetilde{\theta}_n, \widetilde{\theta}_{n+1})$, $\widetilde{\boldsymbol{\theta}} = (\widetilde{\mu}_1, ..., \widetilde{\mu}_n, \widetilde{\mu}_{n+1})$ and $\widetilde{\mathbf{W}}_t = (\widetilde{W}_{1,t}, ..., \widetilde{W}_{n,t}, \widetilde{W}_{I,t})^\intercal$. Here, $\widetilde{\mathbf{\Sigma}}$ denotes the $(n+1)$-dimensional diagonal covariance matrix with its (i, i)-element $\widetilde{\mathbf{\Sigma}}_{i,i} = \widetilde{\sigma}_i^2$ for $i \in \{1, 2, \cdots, n + 1\}$. Notice that the equation in (6) can be converted into[4]

$$dY_t = \Big(r + \gamma\sigma_Y^2 - \frac{f(t)}{l_0 + \int_0^t f(s)ds}\Big)Y_t dt + \sigma_Y Y_t dZ_t, \tag{7}$$

---

[2]See the details in Appendix A.

[3]See the details in Appendix B.

[4]See the details in Appendix C.

where

$$\sigma_Y = \sqrt{\sum_{i=1}^{n+1} \frac{\widetilde{\theta}_i^{\,2}}{\gamma^2}}, \quad \text{and} \quad dZ_t = \frac{\widetilde{\boldsymbol{\theta}}^\intercal}{\gamma\sigma_Y} d\widetilde{\mathbf{W}}_t.$$

Now we can define the liability-adjusted risk-neutral measure, $\mathbb{Q}$, which leads us to

$$dY_t = \Big(r - \frac{f(t)}{l_0 + \int_0^t f(s)ds}\Big) Y_t dt + \sigma_Y Y_t d\widetilde{Z}_t.$$

The drift term shows that, if we think of Y_t as the price process of the underlying asset of a financial derivative under the risk-neutral measure $\mathbb{Q}$, the underlying asset has continuous dividend payments with the rate of $\frac{f(t)}{l_0+\int_0^t f(s)ds}$, or equivalently, $\frac{f(t)}{L_t}$.

Following Kraft and Steffensen (2013), we express the optimal terminal wealth by exploiting the unconstrained optimal funding ratio process, Y_t, and a claim function, g. To construct g, consider the following argument concerning our optimal terminal wealth: if the terminal wealth does not exceed the level of k, it is necessary to prepare for the potential downside risk by purchasing a put option for the terminal wealth. However, the VaR constraint is intrinsically linked to confidence levels, so, if the likelihood of not reaching the predefined threshold is lower than α, there may be no need to prepare for the full extent of the downside risk. Therefore, the fund manager needs to short an additional put option to disregard the significant decrease in the terminal wealth. The argument can be utilized to construct g as

$$g(t, y) = y + (k - y) I_{\{k_\alpha(t) < y < k\}}, \tag{8}$$

where $k_\alpha(t)$ is the level of k where it satisfies $\mathbb{P}(g(t, Y_T) < k) = \alpha$, ($\mathbb{P}(\cdot)$ denotes probability under $\mathbb{P}-$measure) and $I_{\{\cdot\}}$ stands for the indicator function.[5]

Now, let Ψ be the present value of the optimal terminal wealth, then it can be expressed as

$$\Psi(t, y) = E_t^{\mathbb{Q}}[e^{-D(t,r(t))} g(t, Y_T) \big| Y_t = y],$$

where

$$r(t) = r - \frac{f(t)}{l_0 + \int_0^t f(s)ds}, \quad D(t, r(t)) = \int_t^T r(s)ds,$$

and $E_t^{\mathbb{Q}}[\cdot]$ stands for the conditional expectation at time t under $\mathbb{Q}-$measure.[6] On the other hand, if we substitute the value function V for the funding ratio F_t of our constrained problem with a function Φ for the unconstrained optimal funding ratio process Y_t, then the argument in Kraft and Steffensen (2013) gives us[7]

$$\Phi(t, y) = E_t[\widetilde{u}(g(t, Y_T)) \big| Y_t = y],$$
$$\widetilde{u}(x) = \frac{x^{1-\gamma}}{1-\gamma} - \lambda I_{\{x<k\}} \quad \text{for some} \quad \lambda.$$

[5] See the details in Appendix D.

[6] Therefore, the Black-Scholes equation for this case should be

$$\frac{\partial \Psi(t,y)}{\partial t} = r\Psi(t, y) - ry\frac{\partial \Psi(t,y)}{\partial y} - \frac{1}{2}\frac{\boldsymbol{\phi}^\intercal \boldsymbol{\phi}}{\gamma^2} y^2 \frac{\partial^2 \Psi(t,y)}{\partial y^2}, \quad \text{and} \quad \Psi(T, y) = g(y).$$

[7] We aim to utilize the Lagrange multiplier λ to impose lower utility on the function when the constraint is added, compared to the general case where the wealth is under k.

The newly-defined function $\widetilde{u}$ adheres to the intuition that utility decreases significantly if the optimal terminal wealth does not exceed the constraint.

Now the following theorem gives us the relationship between the present value of the optimal terminal wealth, Ψ, and the utility function of the claim, Φ, and their analytic forms.

Theorem 3.1 *The value function V satisfies*

$$V(t, \Psi(t, y)) = \Phi(t, y),$$

Ψ and Φ satisfy the equation in (9), and the optimal proportion $\boldsymbol{\pi}^* = (\pi_{1,t}, \pi_{2,t}, ..., \pi_{n,t}, \pi_{I,t})$ is calculated by the relationship in (10):

$$\frac{\partial \Phi}{\partial y}(t, y) = e^{D(t,\widetilde{r}(t))} y^{-\gamma} \frac{\partial \Psi}{\partial y}(t, y), \tag{9}$$

$$\widetilde{r}(t) = (1-\gamma)\Big(r - \frac{f(t)}{l_0 + \int_0^t f(s)ds} + \frac{\gamma \sigma_Y^2}{2}\Big),$$

$$\boldsymbol{\pi}^*(t, y) = \frac{y \frac{\partial \Psi}{\partial y}(t, y)}{\Psi(t, y)} \mathbf{\Pi}_t^*. \tag{10}$$

Moreover, $\Psi(t, y)$ can be expressed as a closed form of (14) and $\Phi(t, y)$ can be expressed as a closed form of (15), in Appendix E.

Proof. See Appendix E.

4 Numerical Results

4.1 Baseline Parameters

This section examines how the optimal investment strategy is affected by changing market parameters and funding ratio. To pursue our goal, we explore the simplest market structure with one stock index (or a stock portfolio), i.e., $n = 1$, and an inflation-linked index bond (IIB).

We assume that the growth rate of liability is an affine function with respect to the expected inflation rate μ_I:

$$f(t) = (\beta_0 + \beta_1 \mu_I) L_t, \quad \text{thus,} \quad L_t = l_0 e^{(\beta_0 + \beta_1 \mu_I)t},$$

for some constant parameters β_0 and β_1. The affine function we employ is a parsimonious structure which can reflect the relationship between the expected inflation rate and the fund's liability.

The baseline parameters we choose are

$$R = 0.02,\ r = 0.04,\ \mu_1 = 0.08,\ \sigma_1 = 0.2,\ \mu_I = 0.023,\ \sigma_I = 0.05, \gamma = 3,$$
$$\rho \equiv \rho_{1,I} = -0.07,\ \beta_0 = 0.04,\ \beta_1 = 1,\ k = 1,\ \alpha = 0.1, \quad \text{and} \quad T = 5.$$

The parameters mentioned above were set based on previous studies, such as Brennan and Xia (2002) and Kwak and Lim (2014). To ensure all liabilities are paid at the end of the terminal, we set the baseline funding ratio level for the VaR constraint, k, to 1 and the probability of not reaching the goal, α, to 10%. We set the baseline investment period, T, is 5 years, which is normally considered as a mid-term period for pension fund managers.[8]

4.2 Optimal Asset Allocation

(Insert Figure 1 here.)

Based on the baseline parameters, Figure 1 depicts how the optimal investment allocation among two risky assets changes relative to the funding ratio. As the funding ratio climbs, the share allocated to the risky assets declines, in a sufficiently low funding ratio interval. However, at a specific funding ratio, slightly below the target, the allocation to the risky assets stabilizes, slightly surpassing the previous minimum, call it trough.

In times of low funding ratio, the most important for the fund manager is to heighten the portfolio's profitability to meet the target, even at the expense of elevated risk. This necessity arises due to the VaR constraint indispensable for the fund manager, and thus, the demand of the risky assets remains high when the funding ratio is sufficiently low. Approaching the target funding ratio (denoted as $k = 1$ in this context), the fund manager progressively reduces the proportion allocated to both the risky assets. As the goal nears, mitigating downside risk outweighs the pursuit of additional returns for attaining the target securely. Consequently, the allocation to risky assets decreases as the funding ratio escalates.

When the funding ratio slightly dips below the target, the optimal allocation reaches a plateau marginally above the trough. This phenomenon occurs because the downside risk diminishes when the funding ratio hovers slightly beneath the target, implying that achieving the goal becomes comparatively manageable by invest all fund in the risk-free asset. Hence, the optimal investment strategy for the fund manager is to strive for more return.

4.3 Effect of Investment Period

(Insert Figure 2 here.)

Figure 2 shows the optimal investment weights fluctuate across various investment periods contingent upon the funding ratio. Drawing insights from the studies by Marston and Craven (1998) and Edmans (2009), which suggests annual rebalancing as typical for pension funds, each graph portrays the optimal investment strategies for the different investment periods: $T - t = 5, 4, 3, 2, 1$, respectively. It is noteworthy that the graph patterns echo those discussed in Section 4.2.

Furthermore, in Figure 2 we can observe more drastically-changing optimal asset allocation curves for shorter investment periods. It shows that fund managers with less investment period are more likely to use investment strategies that can achieve goals more rapidly and reliably.

[8] For instance, the National Pension Service (NPS) of Korea, which manages the world's third-largest fund, sets a mid-term investment period of 5 years.

4.4 Effect of Expected Inflation Rate and Correlation

(Insert Figure 3 here.)

Figure 3 illustrates the relationship between asset allocation and the funding ratio concerning different levels of the expected inflation rate, μ_I. As the expected inflation rate rises, the optimal allocation of the two risky assets increases. As the expected inflation rate increases, the value of the Inflation-liked Indexed Bonds (IIB) naturally rises. Subsequently, asset allocation towards the IIB expands, and investment in the stock also slightly increases because the stock exhibits a weak negative correlation, $\rho = -0.07$ with the IIB. The negative correlation implies that the two risky assets, the stock and the IIB, have a meaningful hedging relationship with each other, and thus, the investment in the stock should increase when the investment in the IIB increases.

(Insert Figure 4 here.)

Figure 4 depicts the influence of the correlation, ρ, between the two risky assets on the optimal investment strategy. A lower correlation results in a higher investment allocation to both of the two risky assets. This suggests that as the correlation decreases, the hedging effect becomes more significant.

4.5 Sensitivity Analysis

(Insert Table 1 here.)

Table 1 depicts the optimal investment strategies for the different funding ratios, with various parameters. Each row is derived from the baseline parameters, where the parameter of the respective row is adjusted by multiplying it with the given coefficient. Similar to the previous results, the IIB demonstrates a greater sensitivity to changes than the stock. It is also evident that as the fund manager's risk aversion increases, the investment weight of the risky assets decreases, and as the fund's liability increases, the optimal fund manager invests a greater proportion of the fund in the risky assets.

5 Conclusion

This paper explores a liability-driven investment strategy under inflation risk, considering three asset classes: stocks, inflation-linked index bonds, and a risk-free asset. To tackle this issue, we employ the CRRA utility function with respect to the funding ratio. In pursuit of LDI objectives, we impose a VaR constraint on the funding ratio.

Using carefully selected parameters, we show that higher expected inflation rates lead to a greater allocation to both classes of risky assets: stocks and inflation-linked index bonds. This effect becomes more pronounced as the correlation between the two risky asset classes decreases. Additionally, we find that the fund manager's optimal investment strategies should change significantly based on the current funding ratio, with a shorter investment period resulting in greater variability in these strategies. Furthermore, we drive a closed-form optimal investment strategy of our model by extending the option-based approach firstly outlined by Kraft and Steffensen (2013), and this constitutes our technical contribution to the existing literature.

Appendix

A Details for rebuilding the financial market

The objective of this section is to demonstrate that the $(n+1)$ risky assets with correlation can be replaced by $(n+1)$ mutually-uncorrelated risky assets. In other words, for each i, a new risky asset $A_{i,t}$ must be identified that satisfies the following equation. Moreover, we can identify $\boldsymbol{o}_i = (o_{1,i}, ..., o_{n,i}, o_{n+1,i})$ that satisfies the following conditions:

$$\begin{cases} \dfrac{dA_{i,t}}{A_{i,t}} = o_{1,i}\dfrac{dS_{1,t}}{S_{1,t}} + ... + o_{n,i}\dfrac{dS_{n,t}}{S_{n,t}} + o_{n+1,i}\dfrac{dS_{I,t}}{S_{I,t}} \\ o_{1,i} + o_{2,i} + ... + o_{n,i} + o_{n+1,i} = 1 \\ d\langle A_{i,t}, A_{j,t}\rangle = 0 \quad \text{for } i \in \{1, 2, \cdots, n+1\}, i \neq j. \end{cases} \tag{11}$$

First, it is established that $(n+1)$ mutually-uncorrelated Brownian motions can be derived from the $(n+1)$ Brownian motions in our original model through the Gram-Schmidt orthogonalization. Consequently, there exists an $(n+1)\times(n+1)$ lower triangular matrix $\boldsymbol{L}$, which satisfies

$$d\widetilde{\boldsymbol{W}}_t = \boldsymbol{L} d\boldsymbol{W_t}.$$

Furthermore, L satisfies the following equation:

$$d\widetilde{\boldsymbol{W}}_t = \boldsymbol{L\Gamma}^{-1}(\sigma_1 W_{1,t}, ..., \sigma_n W_{n,t}, \sigma_I W_{I,t})^{\intercal}.$$

If the i-th row of $\boldsymbol{L\Gamma}^{-1}$ that satisfies the aforementioned equation is designated as $\widetilde{\boldsymbol{o}}_i = (\widetilde{o}_{1,i}, ..., \widetilde{o}_{n,i}, \widetilde{o}_{n+1,i})$, then all conditions of (11) are satisfied when $\boldsymbol{o}_i = \widetilde{\boldsymbol{o}}_i / \sum_{j=1}^{n+1} \widetilde{o}_{j,i}$.[9]

Finally, define $(n+1)\times(n+1)$ matrix $\boldsymbol{O} = (\boldsymbol{o}_1, ..., \boldsymbol{o}_n, \boldsymbol{o}_{n+1})^{\intercal}$,[10] then if there exist $\boldsymbol{\pi} = (\pi_1, ..., \pi_n, \pi_I)$ and $\widetilde{\boldsymbol{\pi}} = (\widetilde{\pi}_1, ..., \widetilde{\pi}_n, \widetilde{\pi}_{n+1})$ such that satisfy $\widetilde{\boldsymbol{\pi}} = \boldsymbol{O\pi}$ then $\sum_i^n \pi_i \frac{dS_{i,t}}{S_{i,t}} + \pi_I \frac{dS_{I,t}}{S_{I,t}}$ and $\sum_i^{n+1} \widetilde{\pi}_i \frac{dA_{i,t}}{A_{i,t}}$ has same wealth process, which means that we can rebuild the market with independent assets.

B Optimal investment weights and funding ratio process in the rebuilt market

The unconstrained funding ratio process Y_t is expressed as equation (2) in the rebuilt market, as follows:

$$\frac{dY_t}{Y_t} = \Big(\sum_{i=1}^{n+1} \widetilde{\pi}_{i,t}(\widetilde{\mu}_i - r) + r - \frac{f(t)}{l_0 + \int_0^t f(s)ds}\Big)dt + \sum_{i=1}^{n+1} \widetilde{\sigma}_i \widetilde{\pi}_{i,t} d\widetilde{W}_{i,t}$$

where $\widetilde{\pi}_{i,t}$ is proportion of each rebuilt risky assets. Then, value function $\widetilde{V}$ for the investment period $T \in (0, \infty)$, the current time $0 < t < T$, and current wealth x, by controlling her investment

[9] Under the assumption $\rho_{i,j}, \rho_{i,I} \in (-1, 1)$ for all $i, j \in \{1, 2, \cdots, n\}$, $\sum_{j=1}^{n+1} \widetilde{o}_{j,i}$ cannot be zero.

[10] $\boldsymbol{O}$ is a lower triangular matrix with no elements on the diagonal equal to zero. Consequently, since the rank is $(n+1)$, it can be demonstrated that $\boldsymbol{O}$ is indeed an invertible matrix.

strategy $\widetilde{\mathbf{\Pi}}_t = (\widetilde{\pi}_{1,t}, \widetilde{\pi}_{2,t}, ..., \widetilde{\pi}_{n,t}, \widetilde{\pi}_{n+1,t})^\intercal$ is

$$\widetilde{V}(t,x) = \max_{\mathbf{\Pi}_t} E_t\left[\frac{Y_T^{1-\gamma}}{1-\gamma}\middle| X_t = x\right].$$

In this instance, the process outlined in Section 3.2 may be repeated in order to identify the optimal proportion indicated as

$$\widetilde{\mathbf{\Pi}}_t^* = \frac{\widetilde{\mathbf{\Sigma}}^{-1}(\widetilde{\boldsymbol{\mu}} - r\mathbf{1}_{n+1})}{\gamma}.$$

Thus, The unconstrained funding ratio process Y_t can be express as

$$dY_t = \Big(r + \frac{\widetilde{\boldsymbol{\theta}}^\intercal \widetilde{\boldsymbol{\theta}}}{\gamma} - \frac{f(t)}{l_0 + \int_0^t f(s)ds}\Big)Y_t dt + \frac{\widetilde{\boldsymbol{\theta}}^\intercal}{\gamma} Y_t d\widetilde{\mathbf{W}}_t, \quad Y_0 = y_0 > 0.$$

C Construction of σ_Y and Z_t

In order to obtain the value of σ_y, it is necessary to compare the quadratic variation of the equations in (6) and (7). First, the quadratic variation of the equation in (6) is calculated as follows:

$$d\langle Y_t, Y_t\rangle = d\Big\langle \frac{\widetilde{\boldsymbol{\theta}}^\intercal}{\gamma} Y_t d\widetilde{\mathbf{W}}_t, \frac{\widetilde{\boldsymbol{\theta}}^\intercal}{\gamma} Y_t d\widetilde{\mathbf{W}}_t \Big\rangle = \Big(\sum_{i=1}^{n+1} \frac{\widetilde{\theta}_i^2}{\gamma^2}\Big) Y_t^2 dt. \tag{12}$$

Next, the quadratic variation of the equation in (7) is

$$d\langle Y_t, Y_t\rangle = d\langle \sigma_Y Y_t dZ_t, \sigma_Y Y_t dZ_t\rangle = \sigma_Y^2 Y_t^2 dt. \tag{13}$$

The comparison of the two equations in (12) and (13) says that the equations in (6) and (7) are identical when Z_t and σ_Y are given as

$$\sigma_Y = \sqrt{\sum_{i=1}^{n+1} \frac{\widetilde{\theta}_i^{\,2}}{\gamma^2}},$$

$$dZ_t = \frac{\widetilde{\boldsymbol{\theta}}^\intercal}{\gamma \sigma_Y} d\widetilde{\mathbf{W}}_t.$$

D How to find $k_\alpha(t)$

If equation (8) is rephrased in which $g(t,y)$ is defined, it may be expressed as

$$g(t, Y_T) = \begin{cases} Y_T & \text{if } Y_T \geq k \\ k & \text{if } k_\alpha(t) < Y_T < k \\ Y_T & \text{if } Y_T \leq k_\alpha(t) \end{cases}$$

Consequently, the $\mathbb{P}(g(t,Y_T) < k) = \alpha$ is equivalent to $\mathbb{P}(g(t,Y_T) < k_\alpha(t)) = \alpha$ and can be expressed at time t in

$$\mathbb{P}(g(t,Y_T) < k_\alpha(t)) = \frac{1}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{n(Y_t)} \exp\Big\{-\frac{w^2}{2(T-t)}\Big\} dw = \alpha$$

where

$$n(y) = \frac{1}{\sigma_Y}\log\Big(\frac{k_\alpha(t)}{y}\Big) - \Big(\mu_Y - \frac{\sigma_Y^2}{2}\Big)(T-t),$$
$$\mu_Y = r + \gamma\sigma^2 - \frac{f(t)}{l_0 + \int_0^t f(s)ds}.$$

Consequently, $k_\alpha(t)$ may be expressed as

$$k_\alpha(t) = Y_t \exp\Big\{\sigma_Y\Big(\aleph(\alpha) + \Big(\mu_Y - \frac{\sigma_Y^2}{2}\Big)(T-t)\Big)\Big\}.$$

Here, $\aleph(\cdot)$ denotes the inverse cumulative distribution function of a standard normal distribution.

E Proof of Theorem 3.1

Ψ and Φ can be rewritten as

$$\begin{aligned}\Psi(t,y) &= E^{\mathbb{Q}}_{y,t}[e^{-D(t,r(t))}\{Y_T + (k-Y_T)I_{\{Y_T<k\}} - (k_\alpha(t)-Y_T)I_{\{Y_T<k_\alpha(t)\}} - (k-k_\alpha(t))I_{\{Y_T<k_\alpha(t)\}}\}] \\ &= y + Put(t,y,r(t),\sigma_Y,k) - Put(t,y,r(t),\sigma_Y,k_\alpha(t)) - (k-k_\alpha(t))e^{-D(t,r(t))}\mathbb{Q}(Y_T<k_\alpha(t)).\end{aligned} \tag{14}$$

Here, $\mathbb{Q}(\cdot)$ means probability under $\mathbb{Q}-$measure. And

$$\Phi(t,y) = E_{y,t}[\tilde{u}(g(t,Y_T))] = E_{y,t}\left[\frac{1}{1-\gamma}\{Y_T - (k-Y_T)I_{\{k_\alpha(t)<Y_T<k\}}\}^{1-\gamma} - \lambda I_{\{x<k_\alpha(t)\}}\right]$$

where

$$Put(t,y,r(t),\sigma,k) = \aleph(-d_2(t,y,r(t),\sigma,k))ke^{-D(t,r(t))} - \aleph(-d_1(t,y,r(t),\sigma,k))y,$$
$$d_1(t,y,r(t),\sigma,k) = \frac{1}{\sigma\sqrt{T-t}}\left[\log\Big(\frac{y}{k}\Big) + D(t,r(t)) + \frac{\sigma^2(T-t)}{2}\right],$$
$$d_2(t,y,r(t),\sigma,k) = d_1(t,y,r(t),\sigma,k) - \sigma\sqrt{T-t}.$$

Here, $\aleph$ denotes the cumulative distribution function of the standard normal distribution.

Let

$$\tilde{r}(t) = (1-\gamma)\left(r - \frac{f(t)}{l_0+\int_0^t f(s)ds} + \frac{\gamma\sigma_Y^2}{2}\right),$$

then

$$\begin{aligned}\Phi(t,y) =& \frac{e^{D(t,\tilde{r}(t))}}{1-\gamma}E_{y,t}[e^{-D(t,\tilde{r}(t))}\{Y_T^{1-\gamma} - (k^{1-\gamma}-Y_T^{1-\gamma})I_{\{k_\alpha(t)^{1-\gamma}<Y_T^{1-\gamma}<k^{1-\gamma}\}}\}] - \lambda\mathbb{P}(Y_T<k_\alpha(t)) \\ =& \frac{e^{D(t,\tilde{r}(t))}}{1-\gamma}(y^{1-\gamma} + Put(t,y^{1-\gamma},\tilde{r},(1-\gamma)\sigma,k^{1-\gamma}) - Put(t,y^{1-\gamma},\tilde{r},(1-\gamma)\sigma,k_\alpha(t)^{1-\gamma})) \\ &- \frac{e^{D(t,\tilde{r}(t))}}{1-\gamma}(k^{1-\gamma} - k_\alpha(t)^{1-\gamma} + (1-\gamma)\lambda)e^{-D(t,\tilde{r}(t))}\mathbb{P}(Y_T<k_\alpha(t)).\end{aligned}$$

Let

$$\lambda = (k-k_\alpha(t))k_\alpha(t)^{-\gamma} - \frac{k^{1-\gamma}}{1-\gamma} + \frac{k_\alpha(t)^{1-\gamma}}{1-\gamma},$$

then,

$$\begin{aligned}\Phi(t,y) =& \frac{e^{D(t,\tilde{r}(t))}}{1-\gamma}(y^{1-\gamma} + Put(t,y^{1-\gamma},\tilde{r}(t),(1-\gamma)\sigma,k^{1-\gamma}) - Put(t,y^{1-\gamma},\tilde{r}(t),(1-\gamma)\sigma,k_\alpha(t)^{1-\gamma})) \\ &- \frac{e^{D(t,\tilde{r}(t))}}{1-\gamma}k_\alpha(t)^{-\gamma}(k-k_\alpha(t))e^{-D(t,\tilde{r}(t))}\mathbb{P}(Y_T<k_\alpha(t)).\end{aligned} \tag{15}$$

Since we can get

$$\frac{\partial}{\partial y}Put(t, y^{1-\gamma}, \widetilde{r}(t), (1-\gamma)\sigma, k^{1-\gamma}) = (1-\gamma)y^{-\gamma}\frac{\partial}{\partial y}Put(t, y, r(t), \sigma, k),$$

$$e^{-D(t,\widetilde{r}(t))}\frac{\partial}{\partial y}\mathbb{P}(Y_T < k_\alpha(t)) = \frac{y^{-\gamma}}{k_\alpha(t)^{-\gamma}}e^{-D(t,r(t))}\frac{\partial}{\partial y}\mathbb{Q}(Y_T < k_\alpha(t)),$$

we can show that

$$\frac{\partial \Phi}{\partial y}(t, y) = e^{D(t,\widetilde{r}(t))}y^{-\gamma}\frac{\partial \Psi}{\partial y}(t, y).$$

According to Kraft and Steffensen (2013), if the conditions above are satisfied, the optimal investment weight of $A_{i,t}$ can be expressed as

$$\widetilde{\boldsymbol{\pi}}(t, y)^* = \frac{y\frac{\partial \Psi}{\partial y}(t, y)}{\Psi(t, y)}\widetilde{\boldsymbol{\Pi}}_t^*.$$

The final optimal investment weight can be obtained through the last paragraph of A, which shows the relationship between the optimal investment weight of $A_{i,t}$ and the actual optimal investment weight of

$$\boldsymbol{\pi}(t, y)^* = \boldsymbol{O}^{-1}\widetilde{\boldsymbol{\pi}}(t, y)^* = \frac{y\frac{\partial \Psi}{\partial y}(t, y)}{\Psi(t, y)}\boldsymbol{O}^{-1}\widetilde{\boldsymbol{\Pi}}_t^* = \frac{y\frac{\partial \Psi}{\partial y}(t, y)}{\Psi(t, y)}\boldsymbol{\Pi}_t^*.$$

References

Ang, A., B. Chen, and S. M. Sundaresan (2012). Liability driven investment with downside risk. *Available at SSRN*.

Basak, S. and A. Shapiro (2001). Value-at-risk-based risk management: optimal policies and asset prices. *The review of financial studies 14*(2), 371–405.

Brennan, M. J. and Y. Xia (2002). Dynamic asset allocation under inflation. *The journal of finance 57*(3), 1201–1238.

Detemple, J. and M. Rindisbacher (2008). Dynamic asset liability management with tolerance for limited shortfalls. *Insurance: Mathematics and Economics 43*(3), 281–294.

Edmans, A. (2009). Blockholder trading, market efficiency, and managerial myopia. *The journal of finance 64*(6), 2481–2513.

Fischer, S. (1975). The demand for index bonds. *Journal of Political Economy 83*(3), 509–534.

Grossman, S. J. and Z. Zhou (1996). Equilibrium analysis of portfolio insurance. *The Journal of Finance 51*(4), 1379–1403.

Jang, B.-G. and S. Park (2016). Ambiguity and optimal portfolio choice with value-at-risk constraint. *Finance Research Letters 18*, 158–176.

Kraft, H. and M. Steffensen (2013). A dynamic programming approach to constrained portfolios. *European Journal of Operational Research 229*(2), 453–461.

Kwak, M. and B. H. Lim (2014). Optimal portfolio selection with life insurance under inflation risk. *Journal of Banking & Finance 46*, 59–71.

Marston, C. L. and B. M. Craven (1998). A survey of corporate perceptions of short-termism among analysts and fund managers. *The European Journal of Finance 4*(3), 233–256.

Merton, R. C. (1969). Lifetime portfolio selection under uncertainty: The continuous-time case. *The review of Economics and Statistics*, 247–257.

Merton, R. C. (1971). Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory 3*(4), 373–413.

Sharpe, W. F. and L. G. Tint (1990). Liabilities-a new approach. *Journal of Portfolio management 16*(2), 5–10.

Figures and Tables

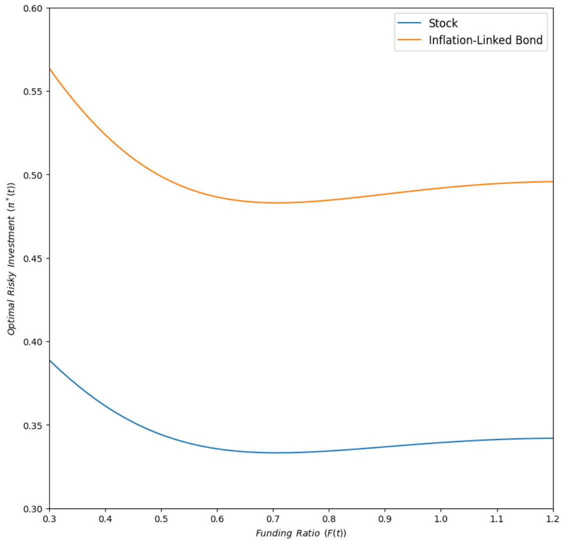

Figure 1: Optimal asset allocation at $t = 0$ with the baseline parameters.

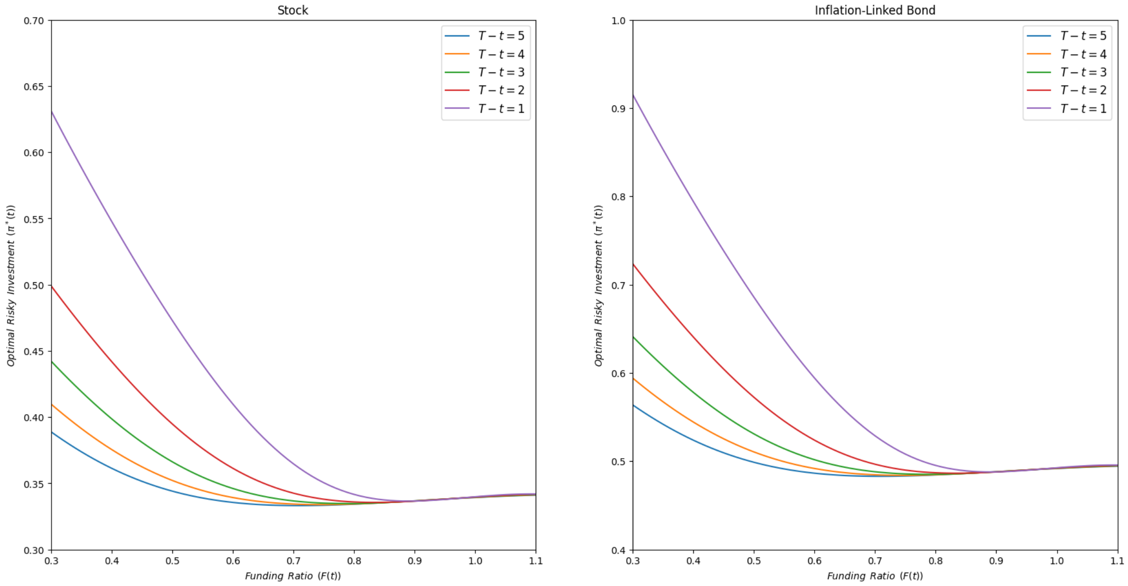

Figure 2: Optimal asset allocation changes according to investment period with the baseline parameters.

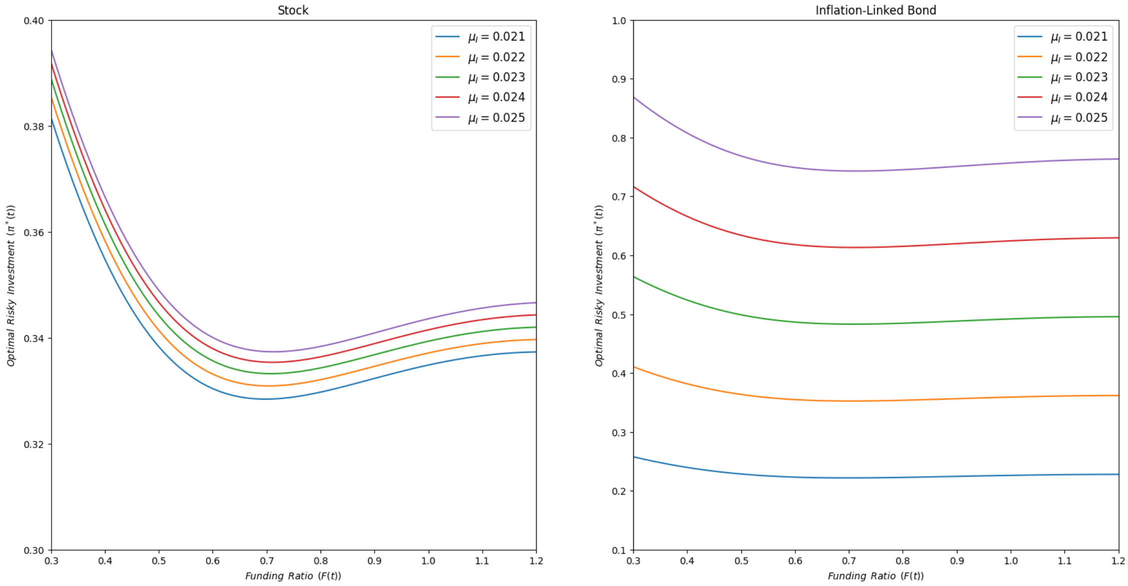

Figure 3: Optimal asset allocation changes according to expected inflation rate with the baseline parameters.

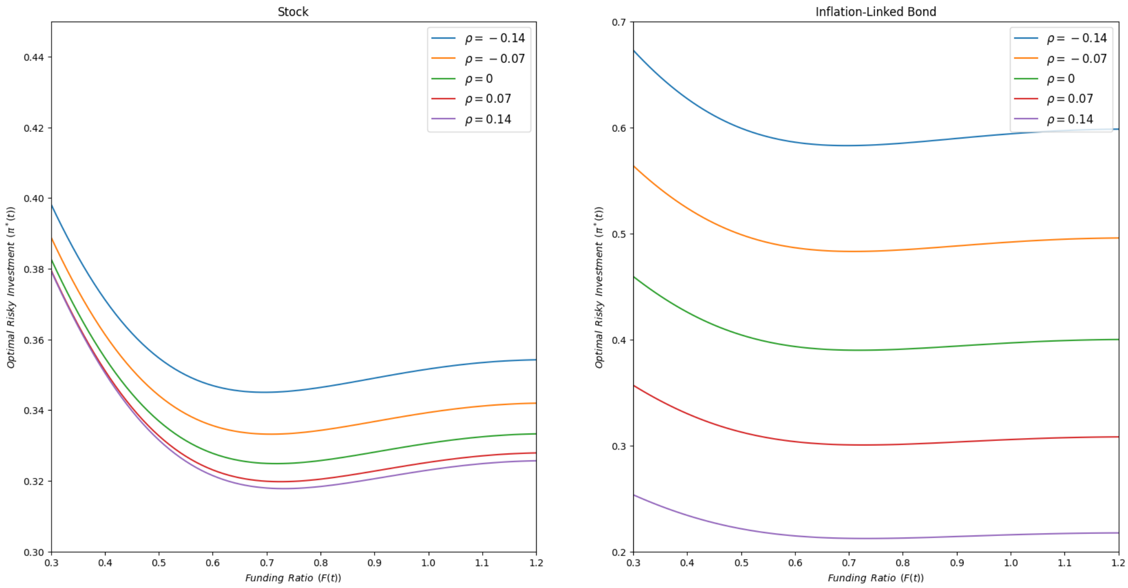

Figure 4: Optimal asset allocation changes according to the correlation between the two risk assets with the baseline parameters

Table 1: Results of the sensitivity analysis

Parameter	Value	Funding Ratio	0.3	0.5	0.7	0.9	1.0	1.2
Baseline		Stock	0.3901	0.3456	0.3349	0.3387	0.3414	0.3443
		IIB	0.7134	0.6320	0.6124	0.6195	0.6244	0.6296
R	1.05	Stock	0.3874	0.3425	0.3314	0.3347	0.3372	0.3396
		IIB	0.4126	0.3648	0.3529	0.3565	0.3591	0.3618
	0.95	Stock	0.3830	0.3335	0.3189	0.3211	0.3233	0.3257
		IIB	0.5552	0.4834	0.4623	0.4654	0.4687	0.4721
γ	1.05	Stock	0.3960	0.3564	0.3492	0.3542	0.3570	0.3599
		IIB	0.5740	0.5166	0.5061	0.5134	0.5175	0.5217
	0.95	Stock	0.3903	0.3449	0.3337	0.3370	0.3395	0.3420
		IIB	0.5657	0.5000	0.4837	0.4885	0.4921	0.4957
α	1.05	Stock	0.3877	0.3434	0.3327	0.3365	0.3392	0.3420
		IIB	0.5619	0.4978	0.4823	0.4878	0.4916	0.4957
	0.95	Stock	0.3992	0.3483	0.3336	0.3363	0.3388	0.3416
		IIB	0.5303	0.4627	0.4432	0.4468	0.4502	0.4539
σ_I	1.05	Stock	0.3790	0.3405	0.3330	0.3373	0.3398	0.3423
		IIB	0.6024	0.5411	0.5292	0.5361	0.5401	0.5441
	0.95	Stock	0.2481	0.2699	0.3074	0.3319	0.3381	0.3420
		IIB	0.3597	0.3913	0.4456	0.4811	0.4901	0.4958
β_0	-1	Stock	0.3151	0.3019	0.3164	0.3328	0.3380	0.3420
		IIB	0.4568	0.4376	0.4587	0.4825	0.4900	0.4958
	0	Stock	0.3890	0.3442	0.3332	0.3368	0.3393	0.3420
		IIB	0.5639	0.4989	0.4830	0.4882	0.4919	0.4957
	1	Stock	0.3044	0.2962	0.3145	0.3325	0.3380	0.3420
		IIB	0.4413	0.4294	0.4559	0.4820	0.4899	0.4958
β_1	-1	Stock	0.3462	0.3190	0.3228	0.3341	0.3384	0.3420
		IIB	0.5018	0.4625	0.4679	0.4844	0.4905	0.4957
	0	Stock	0.3890	0.3442	0.3332	0.3368	0.3393	0.3420
		IIB	0.5639	0.4989	0.4830	0.4882	0.4919	0.4957
	1	Stock	0.3790	0.3405	0.3330	0.3373	0.3398	0.3423
		IIB	0.6024	0.5411	0.5292	0.5361	0.5401	0.5441